

# Radially excited scalar, pseudoscalar, and vector meson nonets in a chiral quark model.

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## Abstract

A chiral Lagrangian containing, besides the usual meson fields, their first radial excitations is constructed. The Lagrangian is derived by bosonization of a Nambu–Jona-Lasinio (NJL) type quark model with separable nonlocal interactions. The nonlocality is described by form factors corresponding to 3-dimensional excited state wave functions. The spontaneous breaking of chiral symmetry is governed by the standard NJL gap equations. A simple  $SU(2) \times SU(2)$  version of the model is used to demonstrate all low-energy theorems to hold valid in the chiral limit.

A more realistic  $U(3) \times U(3)$  model with 't Hooft interaction is constructed to describe the mass spectrum of excited scalar, pseudoscalar, and vector mesons. On the basis of global chiral symmetry, we use the same form factors for the scalar and pseudoscalar mesons. Having fixed the form factor parameters by masses of pseudoscalar mesons, we predict the mass spectrum of scalar mesons. This allows us to interpret experimentally observed scalar, pseudoscalar, and vector meson states as members of quark-antiquark nonets. It is shown that the  $a_0(1450)$ ,  $K_0^*(1430)$ ,  $f_0(1370)$ ,  $f_J(1710)$  scalar meson states are the first radial excitations of the ground states:  $a_0(980)$ ,  $K_0^*(960)$ ,  $f_0(400 - 1200)$ ,  $f_0(980)$ . The weak decay constants  $F_\pi$ ,  $F_{\pi'}$ ,  $F_K$ ,  $F_{K'}$  and the main strong decay widths of the scalar, pseudoscalar, and vector meson nonets are calculated.

# 1. Introduction

The investigation of radial excitations of the scalar, pseudoscalar, and vector meson nonets is of great interest in the hadronic physics. So far, there are questions connected with the experimental and theoretical descriptions of radial excitations of scalar and pseudoscalar mesons. For instance, the experimental data on the excited states of kaons [1] are rare and not reliable enough. There are also problems with interpretation of the experimental data on the scalar and  $\eta$ ,  $\eta'$  mesons. Several years ago, attempts were undertaken to consider the state  $\eta'(1440)$  as a glueball [2].

There is an analogous problem with interpretation of the scalar states  $f_0(1500)$  and  $f_0(1710)$ . Moreover, the experimental status of the lightest scalar isoscalar singlet meson remains unclear. In some papers, the resonance  $f_0(1370)$  was considered as a member of the ground nonet [3, 4], and until 1998 the resonance  $f_0(400 - 1200)$  was not included into the summary tables of PDG review<sup>1</sup> [1].

One will find a problem of the same sort in the case of  $K_0^*$ . The strange meson  $K_0^*(1430)$  seems too heavy to be the ground state: 1 GeV is more characteristic of the ground meson states (see [5, 6]).

Anticipating the results of our review we would like to note that some of these problems were solved in a number of our works which resulted in the present work. From our calculations, for example, we concluded that the states  $\eta(1295)$  and  $\eta(1440)$  can be considered as radial excitations of the ground states  $\eta$  and  $\eta'$ . The estimates of their strong decay widths also confirm our conclusion. Let us note that these meson states are essentially mixed. Our calculations also showed that we can interpret the scalar states  $f_0(1370)$ ,  $a_0(1450)$ ,  $f_0(1710)$ , and  $K_0^*(1430)$  as the first radial excitations of  $f_0(400 - 1200)$ ,  $a_0(980)$ ,  $f_0(980)$ , and  $K_0^*(960)$ .

A theoretical description of radially excited pions poses some interesting challenges. The physics of normal pions is completely governed by the spontaneous breaking of chiral symmetry (SBCS). A convenient way to derive the properties of soft pions is the use of an effective Lagrangian based on a non-linear or linear realization of chiral symmetry [7]. When attempting to introduce higher resonances to extend the effective Lagrangian description to higher energies, one must ensure that the introduction of new degrees of freedom does not spoil the low-energy theorems for pions which are universal consequences of chiral symmetry.

Attempts to describe heavier analogs of the pion, vector mesons, and  $\eta$ ,  $\eta'$  mesons as the radial excitations of well-known ground meson states were made by authors in [2] within the framework of the nonlocal  $^3P_0$  potential quark model. This approach was based on non-relativistic and relativistic quantum mechanics where mesons are treated as bound  $\bar{q}q$  systems.

A useful guideline in the construction of effective meson Lagrangians is the Nambu–Jona-Lasinio (NJL) model that describes SBCS at the quark level with a four-fermion interaction [8, 9, 10, 11]. The bosonization of this model and the derivative expansion of the resulting fermion determinant reproduce the Lagrangian of the linear sigma model that embodies the physics of soft pions as well as higher-derivative terms. With appropriate couplings the model

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<sup>1</sup> However, in earlier editions of PDG the light  $\sigma$  state could still be found; it was excluded later.

allows one to derive also a Lagrangian for vector and axial-vector mesons. This gives not only the correct structure of terms of the Lagrangian as required by chiral symmetry, but also quantitative predictions for the coefficients, such as  $F_\pi$ ,  $F_K$ ,  $g_\pi$ ,  $g_\rho$ , *etc.*

One may, therefore, hope that a suitable generalization of the NJL-model can provide means for deriving an effective Lagrangian including also the excited mesons.

When extending the NJL model to describe radial excitations of mesons, one has to introduce nonlocal (finite-range) four-fermion interactions. Many nonlocal generalizations of the NJL model were proposed, by using either covariant-Euclidean [12] or instantaneous (potential-type) [13, 14] effective quark interactions. These models generally require bilocal meson fields for bosonization, which makes it difficult to perform a consistent derivative expansion leading to an effective Lagrangian.

A simple alternative is to use separable quark interactions. There is a number of advantages of working with that scheme. First, separable interactions can be bosonized by introducing local meson fields, just as the usual NJL-model. One can thus derive an effective meson Lagrangian directly in terms of local fields and their derivatives. Second, separable interactions allow one to introduce a limited number of excited states and only in a given channel.

An interesting method for describing excited meson states in this approximation was proposed in [15]. The authors suggested to consider SBCS in the vicinity of a polycritical point where either all or some of the coupling constants at four-fermion vertices exhibit critical behavior; the critical values of the coupling constants are given by solutions of a set of mass-gap equations. They selected a minimal type of separable four-quark interaction which is most important for the process of SBCS. In this model the form factors are chosen as orthogonal functions, so there is a freedom in their choice up to an arbitrary rotation. All calculations are made in the Euclidean space, by using the approximation of large  $N_c$  and  $\log \Lambda$  where  $\Lambda$  is the ultra-violet cut-off in the model. An interesting result of this approach is that for an arbitrary choice of coupling constants in the vicinity of polycritical point there are multiple solutions with a different critical behavior. Therefore, a problem appears — which of the solutions is realized in nature.

Another advantage of the separable interaction is that it can be defined in Minkowski space in a 3-dimensional (yet covariant) way, with form factors depending only on part of the quark-antiquark relative momentum transverse to the meson momentum [14, 16, 17]. This is essential for a correct description of excited states, since it ensures the absence of spurious relative-time excitations [18]. Finally, as we have shown [17], the form factors defining the separable interaction can be chosen so that the gap equation of the generalized NJL-model coincides with the one of the usual NJL-model, whose solution is a constant (momentum-independent) dynamic quark mass. Thus, in this approach it is possible to describe radially excited mesons above the usual NJL vacuum. Aside from the technical simplification, the latter means that the separable generalization contains all the successful quantitative results of the usual NJL model.

Our work consists of five Sections. In the second Section, we illustrate our method on the basis of a simple  $SU(2) \times SU(2)$  model. Here we prepare grounds for the choice of the

form factors to be used in a more realistic model. It will be shown that we can choose these form factors such that the gap equation conserves its conventional form and has a solution corresponding to a constant constituent quark mass. The quark condensate also does not change after the inclusion of excited states into the model, because the tadpole associated with the excited scalar field is equal to zero (the quark loop with the one excited scalar vertex, vertex with a form factor).

In this Section, we derive an effective chiral Lagrangian describing  $\pi$  and  $\pi'$  mesons from a generalized NJL-model with separable interactions. In Subsection 2.1, we introduce the effective quark interaction in the separable approximation and describe its bosonization. We discuss the choice of form factors necessary to describe excited states. In Subsection 2.2, we solve the gap equation defining the vacuum, derive the effective Lagrangian of the  $0^-$  meson fields, and perform the diagonalization leading to the physical  $\pi$  and  $\pi'$  states. The effective Lagrangian describes the vanishing of the  $\pi$  mass (decoupling of the Goldstone boson) in the chiral limit, while  $\pi'$  remains massive. In Subsection 2.3, we derive the axial vector current of the effective Lagrangian using the Gell-Mann–Levy method and obtain a generalization of the PCAC formula which includes the contribution of  $\pi'$  to the axial current. The leptonic decay constants of the  $\pi$  and  $\pi'$  mesons,  $F_\pi$  and  $F_{\pi'}$ , are discussed in Subsection 2.4. It is shown that  $F_{\pi'}$  vanishes in the chiral limit as expected. In Subsection 2.5, we fix the parameters of the model and evaluate the ratio  $F_{\pi'}/F_\pi$  as a function of the  $\pi'$  mass.

In the third Section, we use the method demonstrated in Section 2 for a realistic description of radially excited states of the scalar, pseudoscalar and vector meson nonets where 't Hooft interaction is included in addition to conventional chirally symmetric four-quark vertices. This allows us to solve the so-called  $U_A(1)$ -problem and describe the masses of ground and excited states of the  $\eta$  and  $\eta'$  mesons.

We take account of the connections of the scalar and vector coupling constants which appeared in this model and an additional renormalization of the pseudoscalar fields connected with the pseudoscalar — axial-vector transitions. For simplicity, we suppose that the masses of  $u$  and  $d$  quarks are equal to each other and take into account only the mass difference between  $(u, d)$  and  $s$  quarks ( $m_u$  and  $m_s$ ). Then, we have in this model six basic parameters:  $m_u$ ,  $m_s$ ,  $\Lambda_3$  (3-dimensional cut-off parameter),  $G$  and  $G_V$  (four-quark coupling constants for the scalar–pseudoscalar coupling ( $G$ ) and for the vector – axial–vector coupling ( $G_V$ )) and constant  $K$  characterizing the 't Hooft interaction. To define these parameters, we use the experimental values: the pion decay constant  $F_\pi = 93$  MeV, the  $\rho$ -meson decay constant  $g_\rho \approx 6.14$  ( $g_\rho^2/(4\pi) \approx 3$ ), the pion mass  $M_\pi \approx 140$  MeV,  $\rho$ -meson mass  $M_\rho = 770$  MeV, the kaon mass  $M_K \approx 495$  MeV, and the mass difference of the  $\eta$  and  $\eta'$  mesons. Using these six parameters, we can describe the masses of four ground meson nonets (pseudoscalar, vector, scalar, and axial–vector) and all the meson coupling constants of strong interactions of mesons with each other and with quarks.

For the investigation of excited states of the mesons it is necessary to consider nonlocal four-quark interactions. In Section 3, it is shown that for the description of excited states of the scalar, pseudoscalar, and vector meson nonets we have to use seven different form factors

in the effective four-quark interactions. Each form factor contains only one free (external) parameter. There are also slope (internal) parameters which are to be fixed by the condition of preserving gap equations in the standard form (see Section 2). We use the same form factors for the scalar and pseudoscalar mesons, which is required by chiral symmetry. This allows us to predict masses of the excited scalar mesons.

In Subsection 3.1, we introduce the effective quark interaction in the separable approximation with 't Hooft terms and describe its bosonization. We discuss the choice of the form factors necessary to describe excited states of the scalar, pseudoscalar, and vector meson nonets. In Subsection 3.2, we derive the effective Lagrangian for the ground and excited states of the strange and isovector scalar and pseudoscalar mesons, and perform the diagonalization leading to the physical ground and excited meson states. In Subsection 3.3, we diagonalize the Lagrangian for the isoscalar scalar and pseudoscalar (ground and excited) mesons and take into account singlet-octet mixing. In Subsection 3.4, we consider vector mesons. In Subsection 3.5, we fix the parameters of the model and evaluate the masses of the ground and excited meson states and the weak decay constants  $F_\pi$ ,  $F_{\pi'}$ ,  $F_K$  and  $F_{K'}$ .

In Section 4, we calculate strong decay widths of excited states of the scalar, pseudoscalar, and vector mesons and compare them with experimental data. In Subsection 4.1, we consider decays of the first radial excitations of  $\pi$ ,  $\rho$  and  $\omega$  meson states. Decays of strange mesons are calculated in Subsection 4.2. Then, in Subsection 4.3, we calculate decay widths of the scalar mesons. Finally, the decay widths of excited  $\eta$  and  $\eta'$  mesons are estimated in Subsection 4.4.

In Section 5 (Conclusion), we briefly discuss our results, give interpretation of the members of meson nonets, and foresee ways of further developing our model.

In Appendix A, we collected some lengthy formulae defining the free part of the effective Lagrangian for isoscalar scalar and pseudoscalar mesons. In Appendix B, we displayed in detail some instructive calculations of strong decay widths of mesons.

## 2. $SU(2) \times SU(2)$ model.

### 2.1 Nambu–Jona-Lasinio model with separable interactions

In this Section, we construct an  $SU(2) \times SU(2)$  NJL-like chiral quark model with quark interaction of the separable type to describe the ground and first radially excited states of pions and  $\sigma$ -mesons. Although, a realistic description of the meson physics requires consideration of a  $U(3) \times U(3)$  version (which we will do in the next Section), we find it instructive to show the basic principles of the model with this simple case. The content of the section corresponds to ref. [17].

In the usual NJL model, SBCS is described by a local (current–current) effective quark interaction. The model is defined by the action

$$S[\bar{\psi}, \psi] = \int d^4x \bar{\psi}(x) (i\not{\partial} - m^0) \psi(x) + S_{\text{int}}, \quad (1)$$

$$S_{\text{int}} = \frac{G}{2} \int d^4x [j_\sigma(x) j_\sigma(x) + j_\pi^a(x) j_\pi^a(x)], \quad (2)$$

where  $j_{\sigma,\pi}(x)$  denote, respectively, the scalar–isoscalar and pseudoscalar–isovector currents of the quark fields ( $SU(2)$ –flavor),

$$j_{\sigma}(x) = \bar{\psi}(x)\psi(x), \quad j_{\pi}^a(x) = \bar{\psi}(x)i\gamma_5\lambda^a\psi(x). \quad (3)$$

The model can be bosonized in a standard way by representing the 4–fermion interaction as a Gaussian functional integral over scalar and pseudoscalar meson fields [8, 9, 10, 11]. Since the interaction, Eq.(2), has represents a product of two local currents, the bosonization is achieved through local meson fields. The effective meson action obtained by integration over quark fields is thus expressed in terms of local meson fields. By expanding the quark determinant in derivatives of the local meson fields, one then derives the chiral meson Lagrangian.

The NJL interaction, Eq.(2), describes only ground–state mesons. To include excited states, one has to introduce effective quark interactions with a finite range. In general, such interactions require bilocal meson fields for bosonization [12, 14]. A possibility to avoid this complication is to use a separable interaction that is still of current–current form, Eq.(2), but allows for nonlocal vertices (form factors) in the definition of quark currents, Eq.(3),

$$\tilde{S}_{\text{int}} = \frac{G}{2} \int d^4x \sum_{i=1}^N \left[ j_{\sigma,i}(x) j_{\sigma,i}(x) + j_{\pi,i}^a(x) j_{\pi,i}^a(x) \right], \quad (4)$$

$$j_{\sigma,i}(x) = \int d^4x_1 \int d^4x_2 \bar{\psi}(x_1) F_{\sigma,i}(x; x_1, x_2) \psi(x_2), \quad (5)$$

$$j_{\pi,i}^a(x) = \int d^4x_1 \int d^4x_2 \bar{\psi}(x_1) F_{\pi,i}^a(x; x_1, x_2) \psi(x_2). \quad (6)$$

Here,  $F_{\sigma,i}(x; x_1, x_2)$ ,  $F_{\pi,i}^a(x; x_1, x_2)$ ,  $i = 1, \dots, N$ , denote a set of nonlocal scalar and pseudoscalar fermion vertices (in general, momentum– and spin–dependent) to be specified below. Upon bosonization Eq.(4) leads to the action

$$\begin{aligned} S_{\text{bos}}[\bar{\psi}, \psi; \sigma_1, \pi_1, \dots, \sigma_N, \pi_N] &= \int d^4x_1 \int d^4x_2 \bar{\psi}(x_1) \left[ (i\cancel{\partial}_{x_2} - m^0) \delta(x_1 - x_2) \right. \\ &\quad \left. + \int d^4x \sum_{i=1}^N \left( \sigma_i(x) F_{\sigma,i}(x; x_1, x_2) + \pi_i^a(x) F_{\pi,i}^a(x; x_1, x_2) \right) \right] \psi(x_2) \\ &\quad - \frac{1}{2G} \int d^4x \sum_{i=1}^N \left( \sigma_i^2(x) + \pi_i^{a2}(x) \right). \end{aligned} \quad (7)$$

It describes a system of local meson fields,  $\sigma_i(x)$ ,  $\pi_i^a(x)$ ,  $i = 1, \dots, N$ , which interact with quarks through nonlocal vertices. We emphasize that these fields are not yet to be associated with physical particles ( $\sigma, \sigma', \dots, \pi, \pi', \dots$ ); physical fields will be obtained after determining the vacuum and diagonalizing the meson effective action.

To define the vertices of Eqs.(5) and (6), and we pass to the momentum representation. Because of the translational invariance, the vertices can be represented as

$$\begin{aligned} F_{\sigma,i}(x; x_1, x_2) &= \int \frac{d^4P}{(2\pi)^4} \int \frac{d^4k}{(2\pi)^4} \exp i \left[ \frac{1}{2}(P+k) \cdot (x-x_1) + \frac{1}{2}(P-k) \cdot (x-x_2) \right] F_{\sigma,i}(k|P), \end{aligned} \quad (8)$$

and similarly for  $F_{\pi,i}^a(x; x_1, x_2)$ . Here  $k$  and  $P$  denote, respectively, the relative and total momentum of a quark–antiquark pair. We take the vertices to depend only on the component of the relative momentum transverse to the total momentum,

$$F_{\sigma,i}(k|P) \equiv F_{\sigma,i}(k_\perp|P), \quad \text{etc.}, \quad k_\perp \equiv k - \frac{P \cdot k}{P^2} P. \quad (9)$$

Here,  $P$  is assumed to be time-like,  $P^2 > 0$ . Equation(9) is a covariant generalization of the condition that the quark–meson interaction is instantaneous in the rest frame of the meson (*i.e.*, the frame in which  $\mathbf{P} = 0$ ). Equation (9) ensures the absence of spurious relative–time excitations and thus leads to a consistent description of excited states<sup>2</sup> [18]. In particular, this framework allows us to use 3–dimensional “excited state” wave functions to model the form factors for radially excited mesons.

The simplest chirally invariant interaction describing scalar and pseudoscalar mesons is defined by spin-independent vertices 1 and  $i\gamma_5\lambda^a$ , respectively. We want to include ground state mesons and their first radial excitation ( $N = 2$ ), and therefore take

$$\left. \begin{array}{l} F_{\sigma,j}(k_\perp|P) \\ F_{\pi,j}^a(k_\perp|P) \end{array} \right\} = \left\{ \begin{array}{l} 1 \\ i\gamma_5\lambda^a \end{array} \right\} \times \Theta(\Lambda_3 - |k_\perp|) f_j(k_\perp), \quad (10)$$

$$f_1(k_\perp) \equiv 1, \quad f_2(k_\perp) = c(1 + d|k_\perp|^2), \quad |k_\perp| \equiv \sqrt{-k_\perp^2}. \quad (11)$$

The step function,  $\Theta(\Lambda_3 - |k_\perp|)$ , is nothing else then a covariant generalization of the usual 3–momentum cutoff of the NJL model in the meson rest frame [14]. The form factor  $f(k_\perp)$  has for  $d < -\Lambda_3^{-2}$  the form of an excited state wave function, with a node in the interval  $0 < |k_\perp| < \Lambda_3$ . Equations (10) and (11) are the first two terms in a series of polynomials in  $k_\perp^2$ ; inclusion of higher excited states would require polynomials of higher degree. Note that the normalization of the form factor  $f(k_\perp)$ , the constant  $c$ , determines the overall strength of the coupling of the  $\sigma_2$  and  $\pi_2$  fields to quarks relative to the usual NJL–coupling of  $\pi_1$  and  $\sigma_1$ .

We remark that the most general vertex could also include spin-dependent structures,  $\not{P}$  and  $\gamma_5\not{P}$ , which in the terminology of the NJL model correspond to the induced vector and axial vector component of  $\sigma$  and  $\pi$  ( $\sigma$ – $\rho$  and  $\pi$ – $A_1$  mixing), respectively. These structures should be considered if vector mesons are included. Furthermore, there could be structures  $\not{k}_\perp$ ,  $\not{P}\not{k}_\perp$  and  $\gamma_5\not{k}_\perp$ ,  $\gamma_5\not{P}\not{k}_\perp$ , respectively, which describe bound states with orbital angular momentum  $L = 1$ . We shall not consider these components here.

With the form factors defined by Eqs.(10) and (11), the bosonized action, Eq.(7), in the momentum representation takes the form

$$\begin{aligned} S_{\text{bos}}[\bar{\psi}, \psi; \sigma_1, \pi_1, \sigma_2, \pi_2] &= \int \frac{d^4k}{(2\pi)^4} \bar{\psi}(k) (\not{k} - m^0) \psi(k) \\ &+ \sum_{j=1}^2 \int \frac{d^4P}{(2\pi)^4} \int_{\Lambda_3} \frac{d^4k}{(2\pi)^4} \bar{\psi}(k + \tfrac{1}{2}P) \left[ \sigma_j(P) + i\gamma_5\lambda^a\pi_j^a(P) \right] f_j(k_\perp) \psi(k - \tfrac{1}{2}P) \end{aligned}$$

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<sup>2</sup>In bilocal field theory, this requirement is usually imposed in the form of the so-called Markov–Yukawa condition of covariant instantaneity of the bound state amplitude [14]. An interaction of the transverse form, Eq.(9), automatically leads to meson amplitudes satisfying the Markov–Yukawa condition.

$$-\frac{1}{2G} \sum_{j=1}^2 \int \frac{d^4 P}{(2\pi)^4} \left( \sigma_j(-P) \sigma_j(P) + \pi_j^a(-P) \pi_j^a(P) \right). \quad (12)$$

Here it is understood that a cutoff in the 3-dimensional transverse momentum is applied to the  $k$ -integral, as defined by the step function of Eq.(10).

## 2.2 Effective Lagrangian for $\pi$ and $\pi'$ mesons

We now want to derive the effective Lagrangian describing physical  $\pi$  and  $\pi'$  mesons. Integrating over the fermion fields in Eq.(12), one obtains the effective action of the  $\sigma_1, \pi_1$ - and  $\sigma_2, \pi_2$ -fields,

$$\begin{aligned} W[\sigma_1, \pi_1, \sigma_2, \pi_2] &= -\frac{1}{2G} \int \frac{d^4 x}{(2\pi)^4} (\sigma_1^2 + \pi_1^{a2} + \sigma_2^2 + \pi_2^{a2}) \\ &\quad - iN_c \text{Tr} \log \left[ i\not{\partial} - m^0 + \sum_{j=1}^2 (\sigma_j + i\gamma_5 \lambda^a \pi_j^a) f_j \right]. \end{aligned} \quad (13)$$

This expression is understood as a shorthand notation for expanding in the meson fields. In particular, we want to derive the free part of the effective action for the  $\pi_1$ - and  $\pi_2$ -fields,

$$W = W^{(0)} + W^{(2)}, \quad (14)$$

$$W^{(2)} = \frac{1}{2} \int \frac{d^4 P}{(2\pi)^4} \sum_{i,j=1}^2 \pi_i^a(P) K_{ij}^{ab}(P) \pi_j^b(P), \quad (15)$$

where we restrict ourselves to timelike momenta,  $P^2 > 0$ . Before expanding in the  $\pi_1$ - and  $\pi_2$ -fields, we must determine the vacuum, *i.e.*, the mean scalar field that arises in the dynamic breaking of chiral symmetry. The mean-field approximation corresponds to the leading order of the  $1/N_c$ -expansion. The mean field is determined by the set of equations

$$\frac{\delta W}{\delta \sigma_1} = -iN_c \text{tr} \int_{\Lambda_3} \frac{d^4 k}{(2\pi)^4} \frac{1}{\not{k} - m^0 + \sigma_1 + \sigma_2 f(k_\perp)} - \frac{\sigma_1}{G} = 0, \quad (16)$$

$$\frac{\delta W}{\delta \sigma_2} = -iN_c \text{tr} \int_{\Lambda_3} \frac{d^4 k}{(2\pi)^4} \frac{f(k_\perp)}{\not{k} - m^0 + \sigma_1 + \sigma_2 f(k_\perp)} - \frac{\sigma_2}{G} = 0. \quad (17)$$

Due to the transverse definition of the interaction, Eq.(9), the mean field inside a meson depends in a trivial way on the direction of the meson 4-momentum,  $P$ . In the following we consider these equations in the rest frame where  $\mathbf{P} = 0, k_\perp = (0, \mathbf{k})$  and  $\Lambda_3$  is the usual 3-momentum cutoff.

In general, the solution of Eqs.(16) and (17) would have  $\sigma_2 \neq 0$ , in which case the dynamically generated quark mass,  $-\sigma_1 - \sigma_2 f(\mathbf{k}) + m^0$ , becomes momentum-dependent. However, if we choose the form factor,  $f(\mathbf{k})$ , such that

$$\begin{aligned} -4mI_1^f &\equiv -iN_c \text{tr} \int_{\Lambda_3} \frac{d^4 k}{(2\pi)^4} \frac{f(\mathbf{k})}{\not{k} - m} = i4N_c m \int_{\Lambda_3} \frac{d^4 k}{(2\pi)^4} \frac{f(\mathbf{k})}{m^2 - k^2} = 0, \\ m &\equiv -\sigma_1 + m^0, \end{aligned} \quad (18)$$



then Eqs.(16) and (17) admit a solution with  $\sigma_2 = 0$  and thus with a constant quark mass,  $m = -\sigma_1 + m^0$ . In this case, Eq.(16) reduces to the usual gap equation of the NJL model,

$$-8mI_1 \equiv -miN_c \int_{\Lambda_3} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2} = \frac{m^0 - m}{G}. \quad (19)$$

Obviously, the condition, Eq.(18), can be fulfilled by choosing an appropriate value of the parameter  $d$  defining the “excited state” form factor, Eq.(11), for given values of  $\Lambda_3$  and  $m$ . Equation(18) expresses the invariance of the usual NJL vacuum,  $\sigma_1 = \text{const.}$ , with respect to variations in the direction of  $\sigma_2$ . In the following, we shall consider the vacuum as defined by Eqs.(18) and (19), *i.e.*, we work with the usual NJL vacuum. We emphasize that this choice is a matter of convenience, not of principle. The qualitative results below could equivalently be obtained with a different choice of form factors; however, in this case one should re-derive all vacuum and ground-state meson properties with the momentum-dependent quark mass. Preserving the NJL vacuum makes formulas below much more transparent and allows us take the parameters fixed in the old NJL model.

With the mean field determined by Eqs.(18) and (19), we now expand the action to quadratic order in the fields  $\pi_1$  and  $\pi_2$ . The quadratic form  $K_{ij}^{ab}(P)$ , Eq.(15), is obtained as

$$\begin{aligned} K_{ij}^{ab}(P) &\equiv \delta^{ab} K_{ij}(P), \\ K_{ij}(P) &= -iN_c \text{tr} \int_{\Lambda_3} \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{\not{k} + \frac{1}{2}\not{P} - m} i\gamma_5 f_i \frac{1}{\not{k} - \frac{1}{2}\not{P} - m} i\gamma_5 f_j \right] - \delta_{ij} \frac{1}{G}, \end{aligned} \quad (20)$$

A graphical representation of the loop integrals in Eq.(20) is given in Fig. 1. The integral is evaluated by expanding in the meson field momentum,  $P$ . To order  $P^2$ , one obtains

$$\begin{aligned} K_{11}(P) &= Z_1(P^2 - M_1^2), & K_{22}(P) &= Z_2(P^2 - M_2^2) \\ K_{12}(P) &= K_{21}(P) = \sqrt{Z_1 Z_2} \Gamma P^2 \end{aligned} \quad (21)$$

where

$$Z_1 = 4I_2, \quad Z_2 = 4I_2^{ff}, \quad (22)$$

$$M_1^2 = Z_1^{-1}(-8I_1 + G^{-1}) = \frac{m^0}{Z_1 G m}, \quad (23)$$

$$M_2^2 = Z_2^{-1}(-8I_1^{ff} + G^{-1}), \quad (24)$$

$$\Gamma = \frac{4}{\sqrt{Z_1 Z_2}} I_2^f. \quad (25)$$

Here,  $I_n$ ,  $I_n^f$ , and  $I_n^{ff}$  denote the usual loop integrals arising in the momentum expansion of the NJL quark determinant, but now with zero, one or two factors  $f(k_\perp)$ , Eq.(11), in the numerator. We may evaluate them in the rest frame,  $k_\perp = (0, \mathbf{k})$ ,

$$I_n^{f..f} \equiv -iN_c \int_{\Lambda_3} \frac{d^4k}{(2\pi)^4} \frac{f(\mathbf{k})..f(\mathbf{k})}{(m^2 - k^2)^n}. \quad (26)$$

The evaluation of these integrals with a 3-momentum cutoff is described, *e.g.*, in ref.[19]. The integral over  $k_0$  is taken by contour integration, and the remaining 3-dimensional integral is regularized by the cutoff. Only the divergent parts are kept; all finite parts are dropped. We point out that the momentum expansion of the quark loop integrals, Eq.(20), is an essential part of this approach. The NJL-model is understood here as a model only for the lowest coefficients of the momentum expansion of the quark loop, but not its full momentum dependence (singularities *etc.*).

Note that a mixing between the  $\pi_1$  and  $\pi_2$  fields occurs only in the kinetic ( $\mathcal{O}(P^2)$ ) terms of Eq.(21), but not in the mass terms. This is a direct consequence of the definition of vacuum by Eqs.(18) and (19), which ensures that the quark loop with one form factor has no  $P^2$ -independent part. The “softness” of the  $\pi_1$ - $\pi_2$  mixing causes the  $\pi_1$ -field to decouple as  $P^2 \rightarrow 0$ . This property is crucial for the appearance of a Goldstone boson in the chiral limit.

To determine the physical  $\pi$ - and  $\pi'$ -meson states, we have to diagonalize the quadratic part of the action, Eq.(15). If one knew the full momentum dependence of the quadratic form, Eq.(21), the masses of physical states would be given as zeros of the determinant of the quadratic form,

$$\det K_{ij}(P^2) = 0, \quad P^2 = M_\pi^2, M_{\pi'}^2. \quad (27)$$

This would be equivalent to the usual Bethe-Salpeter (on-shell) description of bound states: the matrix  $K_{ij}(P^2)$  is diagonalized independently of the respective mass shells,  $P^2 = M_\pi^2, M_{\pi'}^2$  [13, 20, 21]. In our approach, however, we know the quadratic form, Eq.(21), only as an expansion in  $P^2$  at  $P^2 = 0$ . It is clear that the determination of the masses according to Eq.(27) would be incompatible with the momentum expansion, as the determinant involves  $\mathcal{O}(P^4)$ -terms neglected in Eq.(21). To be consistent with the  $P^2$ -expansion, we must diagonalize the kinetic term and the mass term in Eq.(15) simultaneously, with a  $P^2$ -independent transformation of the fields. Let us write Eq.(21) in the matrix form

$$K_{ij}(P^2) = \begin{pmatrix} Z_1 & \sqrt{Z_1 Z_2} \Gamma \\ \sqrt{Z_1 Z_2} \Gamma & Z_2 \end{pmatrix} P^2 - \begin{pmatrix} Z_1 M_1^2 & 0 \\ 0 & Z_2 M_2^2 \end{pmatrix}. \quad (28)$$

The transformation that diagonalizes both the matrices here separately is given by

$$\begin{aligned} \sqrt{Z_1} \pi_1^a &= \frac{\cos \phi}{\sqrt{Z_\pi}} \pi^a + \frac{M_2 \sin \phi}{M_1 \sqrt{Z_{\pi'}}} \pi'^a, \\ \sqrt{Z_2} \pi_2^a &= \frac{M_1 \sin \phi}{M_2 \sqrt{Z_\pi}} \pi^a - \frac{\cos \phi}{\sqrt{Z_{\pi'}}} \pi'^a, \end{aligned} \quad (29)$$

where

$$\tan 2\phi = 2\Gamma \frac{M_1}{M_2} \left(1 - \frac{M_1^2}{M_2^2}\right)^{-1}, \quad (30)$$

$$Z_\pi = \cos^2 \phi + \frac{M_1^2}{M_2^2} \sin^2 \phi + 2\Gamma \frac{M_1}{M_2} \cos \phi \sin \phi, \quad (31)$$

$$Z_{\pi'} = \cos^2 \phi + \frac{M_2^2}{M_1^2} \sin^2 \phi - 2\Gamma \frac{M_2}{M_1} \cos \phi \sin \phi. \quad (32)$$

In terms of the new fields,  $\pi, \pi'$ , the quadratic part of the action, Eq.(15), reads

$$W^{(2)} = \frac{1}{2} \int \frac{d^4 P}{(2\pi)^4} \left[ \pi^a(-P)(P^2 - M_\pi^2) \pi^a(P) + \pi'^a(-P)(P^2 - M_{\pi'}^2) \pi'^a(P) \right]. \quad (33)$$

Here,

$$M_\pi^2 = \frac{M_1^2}{Z_\pi}, \quad M_{\pi'}^2 = \frac{M_2^2}{Z_{\pi'}}. \quad (34)$$

The fields  $\pi$  and  $\pi'$  can thus be associated with physical particles.

Let us now consider the chiral limit, *i.e.*, a vanishing current quark mass,  $m^0 \rightarrow 0$ . From Eqs.(22)–(25) we see that this is equivalent to letting  $M_1^2 \rightarrow 0$ . (Here and in the following, when discussing the dependence of quantities on the current quark mass,  $m^0$ , we keep the constituent quark mass fixed and assume the coupling constant,  $G$ , to be changed in accordance with  $m^0$ , such that the gap equation, Eq.(19) remains fulfilled exactly. In this way, the loop integrals and Eq.(18) remain unaffected by changes of the current quark mass.) Expanding Eqs.(34) in  $M_1^2 \propto m^0$ , one finds

$$M_\pi^2 = M_1^2 + \mathcal{O}(m^4), \quad (35)$$

$$M_{\pi'}^2 = \frac{M_2^2}{1 - \Gamma^2} \left[ 1 + \Gamma^2 \frac{M_1^2}{M_2^2} + \mathcal{O}(M_1^4) \right]. \quad (36)$$

Thus, in the chiral limit the effective Lagrangian, Eq.(33), indeed describes a massless Goldstone pion,  $\pi$ , and a massive particle,  $\pi'$ . Furthermore, in the chiral limit the transformation of the fields, Eq.(29), becomes

$$\begin{aligned} \sqrt{Z_1} \pi_1^a &= \left( 1 - \Gamma^2 \frac{M_1^2}{M_2^2} \right) \pi^a + \frac{\Gamma}{\sqrt{1 - \Gamma^2}} \left( 1 + (1 - \Gamma^2) \frac{M_1^2}{M_2^2} \right) \pi'^a, \\ \sqrt{Z_2} \pi_2^a &= \Gamma \frac{M_1^2}{M_2^2} \pi^a - \frac{1}{\sqrt{1 - \Gamma^2}} \pi'^a. \end{aligned} \quad (37)$$

At  $M_1^2 = 0$  one observes that  $\pi$  has only a component along  $\pi_1$ . This is a consequence of the fact that the  $\pi_1$ – $\pi_2$  coupling in the original Lagrangian, Eq.(21), is of order  $P^2$ . We remark that, although we have chosen to work with the particular choice of excited–state form factor, Eq.(18), the occurrence of a Goldstone boson in the chiral limit in Eq.(13) is general and does not depend on this choice. This may easily be established by using the general gap equations, Eqs.(16) and (17), together with Eq.(20).

## 2.3 The axial current

To describe the leptonic decays of the  $\pi$  and  $\pi'$  mesons, we need the axial current operator. Since our effective action contains, besides the pion, a field describing an “excited state” with the same quantum numbers, it is clear that the axial current of our model is, in general, not carried exclusively by the  $\pi$  field, and is thus not given by the standard PCAC formula. Thus, we must determine the conserved axial current of our model, including the contribution of  $\pi'$ , from first principles.

In general, the construction of the conserved current in a theory with nonlocal (momentum-dependent) interactions is a difficult task. This problem has been studied extensively in the framework of the Bethe–Salpeter equation [22] and various 3-dimensional reductions of it such as the quasipotential and the on-shell reduction [23]. In these approaches, the derivation of the current is achieved by “gauging” all possible momentum dependences of the interaction through minimal substitution, a rather cumbersome procedure in practice. In contrast, in a Lagrangian field theory, a simple method exists to derive conserved currents, the so-called Gell–Mann and Levy method [24], based on the Noether theorem. In this approach, the current is obtained as the variation of the lagrangian with respect to the derivative of a space–time dependent symmetry transformation of the fields. We now show that a suitable generalization of this technique can be employed to derive the conserved axial current of our model with quark–meson form factors depending on the transverse momentum.

To derive the axial current, we start at the quark level. The isovector axial current is the Noether current corresponding to infinitesimal chiral rotations of the quark fields,

$$\psi(x) \rightarrow \left(1 - i\varepsilon^a \frac{1}{2} \lambda^a \gamma_5\right) \psi(x). \quad (38)$$

Following the usual procedure, we consider the parameter of this transformation to be space–time dependent,  $\varepsilon^a \equiv \varepsilon^a(x)$ . However, this dependence should not be completely arbitrary. To describe the decays of  $\pi$  and  $\pi'$  mesons, it is sufficient to know the component of the axial current parallel to the meson 4-momentum,  $P$ . It is easy to see that this component is obtained from chiral rotations whose parameter depends only on the longitudinal part of the coordinate

$$\varepsilon^a(x) \rightarrow \varepsilon^a(x_{||}), \quad x_{||} \equiv \frac{x \cdot P}{\sqrt{P^2}}, \quad (39)$$

since  $\partial_\mu \varepsilon^a(x_{||}) \propto P_\mu$ . In other words, transformations of the form Eq.(39) describe a transfer of the longitudinal momentum to the meson, but not of the transverse momentum. This has an important consequence that the chiral transformation does not change the direction of transversality of the meson–quark interaction, *cf.* Eq.(9). When passing to the bosonized representation, Eq.(7), the transformation of the  $\pi_1, \sigma_1$ - and  $\pi_2, \sigma_2$ -fields induced by Eqs.(38) and (39) is therefore of the form

$$\begin{aligned} \pi_i^a(x) &\rightarrow \pi_i^a(x) + \varepsilon^a(x_{||}) \sigma_i(x), \\ \sigma_i(x) &\rightarrow \sigma_i(x) - \varepsilon^a(x_{||}) \pi_i^a(x). \end{aligned} \quad (i = 1, 2) \quad (40)$$

This follows from the fact that, for a fixed direction of  $P$ , the vertex, Eq.(9), describes an instantaneous interaction in  $x_{||}$ . Thus, the special chiral rotation, Eq.(39), does not mix the components of meson fields coupled to quarks with different form factors.

With the transformation of the chiral fields given by Eqs.(40), the construction of the axial current proceeds exactly as in the usual linear sigma model. We write the variation of the effective action, Eq.(13), in the momentum representation,

$$\delta W = \int \frac{d^4 Q}{(2\pi)^4} \varepsilon^a(Q) D^a(Q), \quad (41)$$

where  $\varepsilon^a(Q) = \tilde{\varepsilon}^a(Q_{||})\delta^{(3)}(Q_{\perp})$  is the Fourier transform of the transformation, Eq.(39), and  $D^a(Q)$  is a function of the fields  $\sigma_i, \pi_i, i = 1, \dots, 2$ , given in the form of a quark loop integral,

$$D^a(Q) = -iN_c \text{tr} \int \frac{d^4k}{(2\pi)^4} \left[ \frac{1}{\not{k} - m} \delta^{ab} + \frac{1}{\not{k} - \frac{1}{2}\not{Q} - m} i\gamma_5 \lambda^a \frac{1}{\not{k} + \frac{1}{2}\not{Q} - m} i\gamma_5 \lambda^b \sigma_1 \right] \times (\pi_1^b(Q) + f(k_{\perp})\pi_2^b(Q)). \quad (42)$$

Here we have used that  $\sigma_2 = 0$  in the vacuum, Eq.(18). Expanding now in the momentum  $Q$ , making use of Eq.(18) and the gap equation, Eq.(19), and setting  $\sigma_1 = -m$  (it is sufficient to consider the symmetric limit,  $m^0 = 0$ ), we get

$$\begin{aligned} D^a(Q) &= -Q^2 m \left[ 4I_2 \pi_1^a(Q) + 4I_2^f \pi_2^a(Q) \right] \\ &= -Q^2 m \left[ Z_1 \pi_1^a(Q) + \sqrt{Z_1 Z_2} \Gamma \pi_2^a(Q) \right]. \end{aligned} \quad (43)$$

The fact that  $D^a(Q^2)$  is proportional to  $Q^2$  is a consequence of the chiral symmetry of the effective action, Eq.(13). Due to this property,  $D^a(Q^2)$  can be regarded as the divergence of a conserved current,

$$A_{\mu}^a(Q) = Q_{\mu} m \left[ Z_1 \pi_1^a(Q) + \sqrt{Z_1 Z_2} \Gamma \pi_2^a(Q) \right]. \quad (44)$$

Equation (44) is the conserved axial current of our model. It is of the usual “PCAC” form, but contains also a contribution of the  $\pi_2$  field. The above derivation was rather formal. However, the result can be understood in simple terms, as is shown in Fig. 2: Both the  $\pi_1$  and  $\pi_2$ -fields couple to the local axial current of the quark field through quark loops; the  $\pi_2$ -field enters the loop with a form factor,  $f(k_{\perp})$ . The necessity to pull out a factor of the meson field momentum (derivative) means that only the  $\mathcal{O}(P^2)$ -parts of the loop integrals,  $I_2$  and  $I_2^f$ , survive, *cf.* Eq.(26). Chiral symmetry ensures that the corresponding diagrams for the divergence of the current have no  $P^2$ -independent part.

The results of this Subsection are an example for the technical simplifications of working with separable quark interactions. The fact that they can be bosonized by local meson fields makes it possible to apply methods of local field theory, such as the Noether theorem, to the meson effective action. Furthermore, we note that the covariant (transverse) definition of the 3-dimensional quark interaction, Eq.(9), is crucial for obtaining a consistent axial current. In particular, with this formulation there is no ambiguity with different definitions of the pion decay constant like with non-covariant 3-dimensional interactions [13].

## 2.4 The weak decay constants of $\pi$ and $\pi'$ mesons

We now use the axial current derived in the previous Subsection to evaluate the weak decay constants of physical  $\pi$  and  $\pi'$  mesons. They are defined by the matrix element of the divergence of the axial current between meson states and vacuum,

$$\langle 0 | \partial^{\mu} A_{\mu}^a | \pi^b \rangle = M_{\pi}^2 F_{\pi} \delta^{ab}, \quad (45)$$

$$\langle 0 | \partial^{\mu} A_{\mu}^a | \pi'^b \rangle = M_{\pi'}^2 F_{\pi'} \delta^{ab}. \quad (46)$$

In terms of the physical fields,  $\pi$  and  $\pi'$ , the axial current takes the form

$$A_\mu^a = P_\mu m \sqrt{Z_1} \left( \pi^a + \Gamma \sqrt{1 - \Gamma^2} \frac{M_1^2}{M_2^2} \pi'^a \right) + \mathcal{O}(M_1^4). \quad (47)$$

Here, we substituted the transformation of the fields, Eq.(37), into Eq.(44). The decay constants of the physical  $\pi$  and  $\pi'$  states are thus given by

$$F_\pi = \sqrt{Z_1} m + \mathcal{O}(M_1^4), \quad (48)$$

$$F_{\pi'} = \sqrt{Z_1} m \Gamma \sqrt{1 - \Gamma^2} \frac{M_1^2}{M_2^2} + \mathcal{O}(M_1^4). \quad (49)$$

The corrections to  $F_\pi$  for excited states are of order  $M_\pi^4$ . Thus, within our accuracy,  $F_\pi$  is identical with the value obtained by the usual NJL model,  $\sqrt{Z_1} m$ , which follows from the Goldberger–Treiman relation at the quark level [8]. On the other hand, the  $\pi'$  decay constant vanishes in the chiral limit  $m^0 \sim M_1^2 \rightarrow 0$ , as expected. We stress that for this property to hold, it is essential to consider the full axial current, Eq.(44), including the contribution of the  $\pi_2$ -component. As can be seen from Eqs.(37) and (44), the standard PCAC formula  $A_\mu^a \propto \partial_\mu \pi_1^a$  would lead to a non-vanishing result for  $F_{\pi'}$  in the chiral limit.

The ratio of the  $\pi'$  to  $\pi$  decay constants can directly be expressed in terms of the physical  $\pi$  and  $\pi'$  masses. From Eqs.(48) and (49) one obtains, using Eqs. (35) and (36),

$$\frac{F_{\pi'}}{F_\pi} = \Gamma \sqrt{1 - \Gamma^2} \frac{M_1^2}{M_2^2} = \frac{\Gamma}{\sqrt{1 - \Gamma^2}} \frac{M_\pi^2}{M_{\pi'}^2}. \quad (50)$$

This is precisely the dependence derived from current algebra considerations in the general “extended PCAC” framework [25]. We note that the same behavior of  $F_{\pi'}$  in the chiral limit is found in models describing chiral symmetry breaking by nonlocal interactions [13, 20].

The effective Lagrangian in a compact way illustrates different consequences of axial current conservation for the pion and its excited state. Both matrix elements of  $\partial_\mu A^\mu$ , Eq.(45) and Eq.(46), must vanish for  $m^0 \rightarrow 0$ . The pion matrix element, Eq.(45), does so by  $M_\pi^2 \rightarrow 0$ , with  $F_\pi$  remaining finite, while for the excited pion matrix element the opposite takes place,  $F_{\pi'} \rightarrow 0$  with  $M_{\pi'}$  remaining finite.

## 2.5 Numerical estimates and conclusions

We can now numerically estimate the excited pion decay constant,  $F_{\pi'}$ , in this model. We take the value of the constituent quark mass  $m = 300 \text{ MeV}$  and fix the 3-momentum cut-off at  $\Lambda_3 = 671 \text{ MeV}$  by fitting the normal pion decay constant  $F_\pi = 93 \text{ MeV}$  in the chiral limit, as in the usual NJL model without excited states, *cf.* [19]. With these parameters one obtains the standard value of the quark condensate,  $\langle \bar{q}q \rangle = -(253 \text{ MeV})^3$ , and  $G = 0.82 m^{-2} = 9.1 \text{ GeV}^{-2}$ ,  $m^0 = 5.1 \text{ MeV}$ . With the constituent quark mass and cutoff fixed, we can determine the parameter  $d$  of the “excited-state” form factor, Eq.(11), from the

condition Eq.(18). We find<sup>3</sup>  $d = -1.83 \Lambda_3^{-2} = -4.06 \text{ GeV}^{-2}$ , corresponding to a form factor  $f(k_\perp)$  with a radial node in the range  $0 \leq |k_\perp| \leq \Lambda_3$ . With this value we determine the  $\pi_1\text{--}\pi_2$  mixing coefficient,  $\Gamma$ , Eq.(25), as

$$\Gamma = 0.41. \quad (51)$$

Note that  $\Gamma$  is independent of the normalization of the form factor  $f(k_\perp)$ , Eq.(11). In fact, the parameter  $c$  enters only into the mass of the  $\pi'$  meson, *cf.* Eqs.(24) and (36); we should not determine its value since the result can directly be expressed in terms of  $M_{\pi'}$ . Thus, Eq.(50) gives

$$\frac{F_{\pi'}}{F_\pi} = 0.45 \frac{M_\pi^2}{M_{\pi'}^2}. \quad (52)$$

For the standard value of the  $\pi'$  mass,  $M_{\pi'} = 1300 \text{ MeV}$ , this comes to  $F_{\pi'} = 0.48 \text{ MeV}$ . The excited pion leptonic decay constant is thus very small, which is a consequence of chiral symmetry. Note that, as opposed to the qualitative results discussed above, the numerical values here depend on the choice of form factor, (see Eq.(18)), and should thus be regarded as a rough estimate.

We remark that the numerical values of the ratio  $F_{\pi'}/F_\pi$  obtained here are comparable to those found in chirally symmetric potential models [20]. However, models describing chiral symmetry breaking by a vector-type confining potential (linear or oscillator) usually underestimate the normal pion decay constant by an order of magnitude [13]. Such models should include a short-range interaction (NJL-type) which is mostly responsible for chiral symmetry breaking.

The small value of  $F_{\pi'}$  does not imply a small width of the  $\pi'$  resonance, since it can decay hadronically, *e.g.*, into  $3\pi$  or  $\rho\pi$ . Such hadronic decays will be investigated in Section 4.

In conclusion, we outlined a simple framework for including radial excitations in an effective Lagrangian description of mesons. The Lagrangian obtained by bosonization of an NJL-model with separable interactions exhibits all qualitative properties expected on general grounds: a Goldstone pion with a finite decay constant, and a massive “excited state” with a vanishing decay constant in the chiral limit. Our model shows in a simple way how chiral symmetry protects the pion from modifications by excited states, which in turn influences the excited states’ contribution to the axial current. These features are general and do not depend on a particular choice of the quark–meson form factor. Furthermore, they are preserved if the derivative expansion of the quark loop is carried to higher orders.

In the investigations described here we strictly kept to an effective Lagrangian approach, where the coupling constants and field transformations are defined at zero momentum. We have no way to check the quantitative reliability of this approximation for radially excited states in the region of  $\sim 1 \text{ GeV}$ , *i.e.*, to estimate the momentum dependence of the coupling constants, within the present model. (For a general discussion of the range of applicability of effective Lagrangians, see [26].) This question can be addressed to generalizations of the

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<sup>3</sup> All parameters will be different when in Section 3 we consider a realistic version of this model. However, the ratio  $d/\Lambda_3$  will be near 2 (its limit as  $\Lambda \rightarrow \infty$ ) and change slightly.

NJL model with quark confinement, which in principle allow both a zero-momentum and an on-shell description of bound states. Recently, first steps were undertaken to investigate the full momentum dependence of correlation functions in an approach of that kind [27].

### 3. $U(3) \times U(3)$ model.

#### 3.1 $U(3) \times U(3)$ chiral Lagrangian with excited meson states and 't Hooft interaction

This Section is devoted to a realistic  $U(3) \times U(3)$  version of the NJL model with nonlocal four-quark interaction (see refs. [28, 29, 30]).

We use a nonlocal separable four-quark interaction of the current-current form which admits nonlocal vertices (form factors) in the quark currents and a pure local six-quark 't Hooft interaction [31, 32]:

$$\mathcal{L}(\bar{q}, q) = \int d^4x \bar{q}(x)(i\not{\partial} - m^0)q(x) + \mathcal{L}_{\text{int}}^{(4)} + \mathcal{L}_{\text{int}}^{(6)}, \quad (53)$$

$$\begin{aligned} \mathcal{L}_{\text{int}}^{(4)} &= \frac{G}{2} \int d^4x \sum_{a=1}^9 \sum_{i=1}^N [j_{S,i}^a(x) j_{S,i}^a(x) + j_{P,i}^a(x) j_{P,i}^a(x)] \\ &\quad - \frac{G_V}{2} \int d^4x \sum_{a=1}^9 \sum_{i=1}^N [j_{V,i}^{a,\mu}(x) j_{V,i,\mu}^a(x) + j_{A,i}^{a,\mu}(x) j_{A,i,\mu}^a(x)], \end{aligned} \quad (54)$$

$$\mathcal{L}_{\text{int}}^{(6)} = -K [\det [\bar{q}(1 + \gamma_5)q] + \det [\bar{q}(1 - \gamma_5)q]] \quad (55)$$

where  $\mathcal{L}_{\text{int}}^{(4)}$  is the  $U(3) \times U(3)$  chirally symmetric four-quark interaction Lagrangian and  $\mathcal{L}_{\text{int}}^{(6)}$  contains the symmetry breaking 't Hooft terms. Here,  $m^0$  is the current quark mass matrix  $m^0 = \text{diag}(m_u^0, m_d^0, m_s^0)$  ( $m_u^0 \approx m_d^0$ ) and  $j_{U,i}^a$  with  $U = (S, P, V, A)$  denotes the scalar, pseudoscalar, vector, and axial-vector quark currents

$$j_{S(P),i}^a(x) = \int d^4x_1 d^4x_2 \bar{q}(x_1) F_{S(P),i}^a(x; x_1, x_2) q(x_2), \quad (56)$$

$$j_{V(A),i}^{a,\mu}(x) = \int d^4x_1 d^4x_2 \bar{q}(x_1) F_{V(A),i}^{a,\mu}(x; x_1, x_2) q(x_2) \quad (57)$$

where  $F_{S(P),i}^a(x; x_1, x_2)$  are the scalar (pseudoscalar) and  $F_{V(A),i}^{a,\mu}(x; x_1, x_2)$  the vector and axial-vector nonlocal quark vertices. The index  $a = 1, \dots, 9$  denotes the basis elements  $\tau^a$  of  $U(3)$  flavor group. Our choice is slightly different from the Gell-Mann  $\lambda$  matrices

$$\begin{aligned} \tau_i &= \lambda_i \quad (i = 1, \dots, 7), \quad \tau_8 = (\lambda_0 + \lambda_8)/\sqrt{3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ \tau_9 &= (-\lambda_0 + \sqrt{2}\lambda_8)/\sqrt{3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}, \end{aligned} \quad (58)$$



but this choice is more convenient when a singlet-octet mixing appears due to the 't Hooft terms.

In the original formulation of the NJL model with 't Hooft interaction, the 't Hooft terms are represented by six-fermion vertices. In this form the Lagrangian is not ready for the bosonization procedure, we should proceed to. An appropriate way to circumvent this drawback is to come to an equivalent form of the quark Lagrangian that contains only four-quark vertices as it was done, *e.g.*, in refs. [4, 31]. Therein, the effective four-quark interaction is deduced by integrating out a quark loop at each six-quark vertex. Thus, from  $\mathcal{L}_{\text{int}}^{(6)}$  the four-quark part  $\mathcal{L}_{\text{int}}^{(4)}$  acquires an additional contribution which in the one-loop approximation looks as follows:

$$\begin{aligned}
L_{\text{tH}}^{(4)} = & 4K \int d^4x \left\{ \sum_{a=1}^3 m_s I_1(m_s) \left[ (\bar{q}(x) i\gamma_5 \tau^a q(x))^2 - (\bar{q}(x) \tau^a q(x))^2 \right] \right. \\
& + \sum_{a=4}^7 m_u I_1(m_u) \left[ (\bar{q}(x) i\gamma_5 \tau^a q(x))^2 - (\bar{q}(x) \tau^a q(x))^2 \right] \\
& + m_s I_1(m_s) \left[ (\bar{q}(x) \tau^8 q(x))^2 - (\bar{q}(x) i\gamma_5 \tau^8 q(x))^2 \right] \\
& - 2\sqrt{2} m_u I_1(m_u) \left[ (\bar{q}(x) \tau^8 q(x)) (\bar{q}(x) \tau^9 q(x)) \right. \\
& \left. \left. - (\bar{q}(x) i\gamma_5 \tau^8 q(x)) (\bar{q}(x) i\gamma_5 \tau^9 q(x)) \right] \right\}. \tag{59}
\end{aligned}$$

In our model the 't Hooft interaction is local with respect to its instantaneous origin. Finally, we have<sup>4</sup>:

$$\begin{aligned}
\mathcal{L}(\bar{q}, q) = & \int d^4x \bar{q}(x) (i\cancel{\partial} - m^0) q(x) \\
& + \frac{1}{2} \int d^4x \sum_{a=1}^9 \sum_{b=1}^9 [G_{ab}^{(-)} j_{S,1}^a(x) j_{S,1}^b(x) + G_{ab}^{(+)} j_{P,1}^a(x) j_{P,1}^b(x)] \\
& + \frac{G}{2} \int d^4x \sum_{a=1}^9 \sum_{i=2}^N [j_{S,i}^a(x) j_{S,i}^a(x) + j_{P,i}^a(x) j_{P,i}^a(x)] \\
& - \frac{G_V}{2} \int d^4x \sum_{a=1}^9 \sum_{i=1}^N [j_{V,i}^{a,\mu}(x) j_{V,i,\mu}^a(x) + j_{A,i}^{a,\mu}(x) j_{A,i,\mu}^a(x)]. \tag{60}
\end{aligned}$$

where

$$\begin{aligned}
G_{11}^{(\pm)} &= G_{22}^{(\pm)} = G_{33}^{(\pm)} = G \pm 4K m_s I_1(m_s), \\
G_{44}^{(\pm)} &= G_{55}^{(\pm)} = G_{66}^{(\pm)} = G_{77}^{(\pm)} = G \pm 4K m_u I_1(m_u), \\
G_{88}^{(\pm)} &= G \mp 4K m_s I_1(m_s), \quad G_{99}^{(\pm)} = G, \quad G_{89}^{(\pm)} = G_{98}^{(\pm)} = \pm 4\sqrt{2} K m_u I_1(m_u), \\
G_{ab} &= 0 \quad (a \neq b, \ a, b = 1, \dots, 7). \tag{61}
\end{aligned}$$

The model thus formulated can be bosonized in a standard way by introducing auxiliary boson fields  $\sigma_i^a(x), \varphi_i^a(x), V_i^\mu(x), A_i^\mu(x)$  with quantum numbers of the quark currents  $j_{S(P),i}^a(x)$

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<sup>4</sup> It should be noted that SBOS is already taken into account in the effective four-fermion vertices. Therefore, the effective four fermion Lagrangian is no longer chirally invariant. However, in its original form the chiral invariance is present if we exclude 't Hooft terms. This fact has some consequences which we use later, for instance, we choose the same form factors both for scalars and pseudoscalars.

and  $j_{V(A),i}^{a,\mu}$ , and then integrating over the quark degrees of freedom. The result is a meson effective Lagrangian which, after all, is a functional of scalar, pseudoscalar, vector, and axial-vector meson fields. In the case of an ordinary (local) NJL model, this procedure would give us the well-known linear realization of the chiral Lagrangian. When original four-quark vertices of the separable type contain form factors, the bosonization gives rise to a meson effective Lagrangian for the ground state and a number (in general infinite) of radially excited meson fields. These fields have the same quantum numbers and therefore should be interpreted as “radial” excitations.

The effective four-quark representation of the Lagrangian with 't Hooft interaction requires careful treatment. It is not equivalent to the original form in all aspects. For example, the gap equations derived from the effective four-quark form of the Lagrangian do not reproduce those obtained from the original form (with six-quark vertices). A kind of double counting takes place here, which leads to wrong gap equations (for a correct derivation of gap equations, see [31]). But for the mass spectra and meson-meson coupling constants in the one-loop approximation, everything works well.

In the one-loop approximation, the bosonized Lagrangian has the following form:

$$\begin{aligned}
L_{\text{bos}}(\bar{q}, q; \sigma, \varphi, V, A) = & \int d^4x_1 \int d^4x_2 \bar{q}(x_1) \left[ (i\not{\partial}_{x_2} - m^0) \delta(x_1 - x_2) \right. \\
& + \int d^4x \sum_{i=1}^N \sum_{a=1}^9 \left( \sigma_i^a(x) F_{\sigma,i}^a(x; x_1, x_2) + \varphi_i^a(x) F_{\varphi,i}^a(x; x_1, x_2) \right. \\
& \left. \left. + V_{i,\mu}^a(x) F_{V,i}^{a,\mu}(x; x_1, x_2) + A_{i,\mu}^a(x) F_{A,i}^{a,\mu}(x; x_1, x_2) \right) \right] q(x_2) \\
& - \sum_{a=1}^9 \int d^4x \left[ \frac{1}{2} \left( (G^{(-)})_{ab}^{-1} \sigma_1^a(x) \sigma_1^b(x) + (G^{(+)})_{ab}^{-1} \varphi_1^a(x) \varphi_1^b(x) \right) \right. \\
& \left. - \frac{1}{2G_V} \left( (V_1^{a,\mu}(x))^2 + (A_1^{a,\mu}(x))^2 \right) \right] \\
& - \int d^4x \sum_{i=2}^N \left[ \frac{1}{2G} \left( (\sigma_i^a(x))^2 + (\phi_i^a(x))^2 \right) - \frac{1}{2G_V} \left( (V_i^{a,\mu}(x))^2 + (A_i^{a,\mu}(x))^2 \right) \right]. \quad (62)
\end{aligned}$$

This Lagrangian describes a system of local meson fields,  $\sigma_i^a(x)$ ,  $\phi_i^a(x)$ ,  $V_i^{a,\mu}(x)$ ,  $A_i^{a,\mu}(x)$ ,  $i = 1, \dots, N$ , which interact with quarks through nonlocal vertices. These fields are not yet to be associated with physical particles, to be obtained after determining the vacuum and diagonalizing the meson effective Lagrangian.

In general, the model admits as many excited states as one wishes. But for a realistic description of very heavy mesons (2 GeV and more) the model seems not reliable because it is constructed for low energies. So we intended here to consider a minimal version of the model, restricting ourselves to  $N = 2$ , which is necessary for the description of ground states and first radial excitations of mesons.

To describe the ground and first radially excited states of mesons, we take the form factors in the momentum representation as follows:

$$F_{S,j}^a(\mathbf{k}) = \tau^a f_{\sigma,j}^a, \quad F_{P,j}^a = i\gamma_5 \tau^a f_{\varphi,j}^a, \quad (63)$$

$$F_{V,j}^{a,\mu}(\mathbf{k}) = \gamma^\mu \tau^a f_{V,j}^a, \quad F_{A,j}^{a,\mu} = \gamma_5 \gamma^\mu \tau^a f_{A,j}^a, \quad (64)$$

$$f_{U,1}^a \equiv 1, \quad f_{U,2}^a \equiv f_a^U(\mathbf{k}) = c_a^U(1 + d_a \mathbf{k}^2), \quad (65)$$

where  $U = (\sigma, \varphi, V, A)$ . Here, we consider the form factors in the rest frame of mesons (see Section 2). After bosonization in the one-loop approximation, we get

$$\begin{aligned} L_{\text{bos}}(\sigma, \varphi, V, A) = & - \sum_{a,b=1}^9 \int d^4x \left[ \frac{1}{2} \left( (G^{(-)})_{ab}^{-1} \bar{\sigma}_1^a(x) \bar{\sigma}_1^b(x) + (G^{(+)})_{ab}^{-1} \varphi_1^a(x) \varphi_1^b(x) \right) \right. \\ & - \frac{1}{2G_V} \left( (V_1^{a,\mu}(x))^2 + (A_1^{a,\mu}(x))^2 \right) \Big] \\ & - \sum_{a=1}^9 \int d^4x \left[ \frac{1}{2G} \left( (\sigma_2^a(x))^2 + (\phi_2^a(x))^2 \right) - \frac{1}{2G_V} \left( (V_2^{a,\mu}(x))^2 + (A_2^{a,\mu}(x))^2 \right) \right] \\ & - i \text{Tr} \ln \left[ 1 + \frac{1}{i \not{\partial} - m} \sum_{j=1}^2 \sum_{a=1}^9 (\sigma_j^a + \varphi_j^a + V_j^{a,\mu} \gamma_\mu + A_j^{a,\mu} \gamma_5 \gamma_\mu) f_j^a \tau_a \right] \end{aligned} \quad (66)$$

At the beginning of this Section, we have already mentioned that there is a danger of double counting when deriving gap equations. The double counting surely takes place if one tries to obtain the gap equations by naïvely varying the Lagrangian (66) over  $\sigma_1^a$ . However, correct equations for  $\sigma_2^a$  can be obtained in this way. It is due to the fact that the 't Hooft interaction is local.

The gap equations for  $\sigma_1^a$  can be deduced from the Dyson-Schwinger equation. We will not discuss the details of finding its solution but refer the reader, *e. g.*, to paper [31]. Here we present just the result that is a slight modification of the equations obtained in ref. [31].

$$m_u^0 = m_u [1 - 8G_{88}^{(-)}(I_1(m_u) + I_1^{fuu}(m_u)f_2^8)], \quad (67)$$

$$m_s^0 = m_s [1 - 8G_{99}^{(-)}(I_1(m_s) + I_1^{fss}(m_s)f_2^9)]. \quad (68)$$

There  $m_a^0$  and  $m_a$  ( $a = u, d, s$ ) are the current and constituent quark masses, respectively. The difference between Eqs.(67),(68) and those given in ref.[31] is the presence of  $I_1^f(m_u)$ , tadpoles with form factors absent in local NJL.

The constituent quark masses appear, as usual, due to non-zero vacuum expectations of  $\sigma^a$ , according to the equations

$$\langle \bar{\sigma}^8 \rangle_0 = m_u^0 - m_u, \quad \langle \bar{\sigma}^9 \rangle_0 = m_s^0 - m_s. \quad (69)$$

We use them in the gap equations for excited meson states. The fields  $\bar{\sigma}^a$  require redefinition which consists in subtracting their vacuum expectation values:

$$\sigma^8 = \bar{\sigma}^8 - \langle \bar{\sigma}^8 \rangle_0, \quad \sigma^9 = \bar{\sigma}^9 - \langle \bar{\sigma}^9 \rangle_0. \quad (70)$$

Now we stop discussing the gap equation for the ground fields and turn our attention to those for radially excited meson states. As it was said above, the correct gap equations for

radially excited meson states can be obtained by calculating the first derivative of Lagrangian (66) with respect to  $\sigma_2^a$ , which gives

$$\langle \frac{\delta L}{\delta \sigma_2^a} \rangle_0 = -iN_c \text{tr} \int_{\Lambda_3} \frac{d^4 k}{(2\pi)^4} \frac{f^a(\mathbf{k})}{(\not{k} - m + \langle \sigma_2^a \rangle_0 \tau_a f^a(\mathbf{k}))} - \frac{\langle \sigma_2^a \rangle_0}{G} = 0. \quad (71)$$

This equation always admits the trivial solution  $\langle \sigma_2^a \rangle = 0$ . Despite the fact that nontrivial solutions are possible, we assume that the vacuum expectations for radially excited meson states are equal to zero and therefore do not change the quark condensate. Thus, we obtain the condition

$$-iN_c \text{tr} \int_{\Lambda_3} \frac{d^4 k}{(2\pi)^4} \frac{f^a(\mathbf{k})}{(\not{k} - m)} = 0. \quad (72)$$

Equation (72) is written in the matrix form. In the isotopic symmetry, Eq.(72) gives two conditions on the form factors  $f^a(\mathbf{k})$  which can be written in our notation as follows:

$$I_1^{f_{uu}}(m_u) = 0, \quad (73)$$

$$I_1^{f_{ss}}(m_s) = 0. \quad (74)$$

These conditions essentially simplify the calculation of the meson mass spectra. In particular, they provide a diagonal form for the  $(\sigma_i^a)^2$  and  $(\varphi_i^a)^2$  mass terms of the meson Lagrangian, however, not for all contributions. To ensure that no terms like  $\sigma_1^a \sigma_2^a$  or  $\varphi_1^a \varphi_2^a$  for strange mesons come from the one-loop quark integrals, we must impose, in addition to Eqs.(73) and (74), another condition

$$I_1^{f_{us}}(m_u) + I_1^{f_{us}}(m_s) = 0. \quad (75)$$

Conditions (73), (74), and (75) provides orthogonality of the ground ( $i = 1$ ) and excited ( $i = 2$ ) meson states in the low energy limit  $P^2 \rightarrow 0$  (see Section 2) when  $\varphi_1^a$  become Goldstone bosons.

Now let us remind how we fix the basic parameters in the usual NJL model without excited states of mesons [10].

To obtain correct coefficients of kinetic terms of mesons in the quark-one-loop approximation, we have to make the renormalization of the meson fields

$$\sigma_a = g_\sigma^a \sigma_a^r, \quad \varphi_a = g_\sigma^a \varphi_a^r, \quad V_a^\mu = \frac{g_V^a}{2} V_a^{\mu,r}, \quad A_a^\mu = \frac{g_V^a}{2} A_a^{\mu,r}, \quad (76)$$

where

$$g_\sigma^{a,j} = [4I_2(m_i, m_j)]^{-1/2}, \quad I_2(m_i, m_j) = -iN_c \int_{\Lambda_3} \frac{d^4 k}{(2\pi)^4} \frac{1}{(m_i^2 - k^2)(m_j^2 - k^2)}, \quad (77)$$

$$g_V^a = \sqrt{6} g_\sigma^a. \quad (78)$$

After taking account of the pseudoscalar – axial–vector transitions ( $\varphi_a \rightarrow A_a$ ), the additional renormalization of the pseudoscalar fields

$$g_\sigma^a = Z_a^{-\frac{1}{2}} g_\sigma^a, \quad (79)$$

appears, where  $Z_\pi = 1 - 6m_u^2/M_{a_1}^2 \approx 0.7$  for pions. ( $M_{a_1} = 1.23$  GeV is the mass of the axial-vector  $a_1$  meson, [1],  $m_u = 280$  MeV (see below and [10]) . We assume that  $Z_a \approx Z_\pi \approx 0.7$  for any  $a$ .

After these renormalizations the part of the Lagrangian describing the ground states of mesons takes the form

$$L(\sigma, \phi, V, A) = -\frac{1}{2}((G^{(-)})_{ab}^{-1}g_\sigma^a g_\sigma^b \sigma_a \sigma_b + (G^{(+)}_{ab})^{-1}g_\phi^a g_\phi^b \phi_a \phi_b) - \frac{g_V^2}{2G_V}(V_a^2 + A_a^2) - iN_c \text{Tr} \log \left[ i\not{\partial} - m + \left( g_\sigma^a \sigma_a + i\gamma_5 g_\phi^a \phi_a + \frac{g_V^a}{2}(\gamma_\mu V_a^\mu + \gamma_5 \gamma_\mu A_a^\mu) \right) \tau^a \right]. \quad (80)$$

for simplicity we omitted the index  $r$  of meson fields.

Lagrangian (80) in the one-loop approximation results in, the following expressions for the meson masses [10]

$$M_\pi^2 = g_\pi^2 \left[ \frac{1}{G_\pi} - 8I_1(m_u) \right] = \frac{g_\pi^2 m_u^0}{G_\pi m_u}, \quad g_\pi^2 = \frac{1}{4ZI_2(m_u, m_u)}, \quad (81)$$

$$M_K^2 = g_K^2 \left[ \frac{1}{G_K} - 4(I_1(m_u) + I_1(m_s)) \right] + Z^{-1}(m_s - m_u)^2, \quad g_K^2 = \frac{1}{4ZI_2(m_u, m_s)}, \quad (82)$$

$$G_\pi = G_{11}^{(+)}, \quad G_K = G_{44}^{(+)}, \quad (83)$$

$$M_{88}^{(-)} = g_{\eta_u}^2 \left( (G^{(+)}_{88})^{-1} - 8I_1(m_u) \right), \quad (84)$$

$$M_{99}^{(-)} = g_{\eta_s}^2 \left( (G^{(+)}_{99})^{-1} - 8I_1(m_s) \right), \quad (85)$$

$$M_{89}^{(-)} = g_{\eta_u} g_{\eta_s} \left( (G^{(+)}_{89})^{-1} \right). \quad (86)$$

$$M_{(\eta, \eta')}^2 = \frac{1}{2} \left[ M_{88}^{(-)} + M_{99}^{(-)} \mp \sqrt{(M_{88}^{(-)} - M_{99}^{(-)})^2 + 4(M_{89}^{(-)})^2} \right], \quad (87)$$

$$M_\rho^2 = \frac{g_\rho^2}{4G_V} = \frac{3}{8G_V I_2(m_u, m_u)}, \quad M_\varphi^2 = M_\rho^2 \frac{I_2(m_u, m_u)}{I_2(m_s, m_s)}, \quad (88)$$

$$M_{K^*}^2 = M_\rho^2 \frac{I_2(m_u, m_u)}{I_2(m_u, m_s)} + \frac{3}{2}(m_s - m_u)^2. \quad (89)$$

Now let us fix our basic parameters. For that we use six experimental values [8, 10, 32]:

- 1) The pion decay constant  $F_\pi = 93$  MeV .
- 2) The  $\rho$ -meson decay constant  $g_\rho \approx 6.14$ . Then from the Goldberger-Treiman identity we obtain

$$m_u = F_\pi g_\pi \quad (90)$$

and from Eqs. (78) and (79) we get

$$g_\pi = \frac{g_\rho}{\sqrt{6Z}}, \quad m_u = \frac{F_\pi g_\rho}{\sqrt{6Z}}, \quad m = 280 \text{ MeV}. \quad (91)$$

From Eqs. (77) and (78) we can obtain (see [19])

$$I_2(m_u, m_u) = \frac{3}{2g_\rho^2}, \quad \Lambda_3 = 1.03 \text{ GeV}. \quad (92)$$

3)  $M_\pi = 135$  MeV, the Eq.(81) gives  $G_\pi$ .

4)  $M_\rho = 770$  MeV, the Eq.(82) gives  $G_V$ .

5)  $\left. \begin{array}{l} M_K \approx 495 \text{ MeV}, \\ M_{\eta'}^2 - M_\eta^2 \end{array} \right\} \text{fix } K \text{ and } m_s.$

Then the masses of  $\eta$ ,  $\eta'$ ,  $K^*$ ,  $\varphi$ , and scalar mesons can be calculated with a satisfactory accuracy (see [32]).

We can calculate the values of  $F_K$  and all the coupling constants of strong interactions of scalar, pseudoscalar, vector, and axial-vector mesons with each other and with quarks, and describe the main decays of these mesons (see [10, 32]).

Further, when the radial excitations are included, the parameters will be shifted because of changing the mass formulae. However,  $m_u$  and  $\Lambda_3$  will be the same as they are now. Their numerical values will be calculated in Subsec. 3.5.

### 3.2 The masses of isovector and strange scalar and pseudoscalar mesons (ground and excited states)

After bosonization, the part of Lagrangian (66) describing the isovector and strange scalar and pseudoscalar mesons takes the form

$$\begin{aligned} \mathcal{L}(a_{0,1}, K_{0,1}^*, \pi_1, K_1, a_{0,2}, K_{0,2}^*, \pi_2, K_2) = & -\frac{a_{0,1}^2}{2G_{a_0}} - \frac{K_{0,1}^{*2}}{G_{K_0^*}} - \frac{\pi_1^2}{2G_\pi} - \frac{K_1^2}{G_K} - \\ & \frac{1}{2G}(a_{0,2}^2 + 2(K_{0,2}^*)^2 + \pi_2^2 + 2K_2^2) - \\ & iN_c \text{Tr} \ln \left[ 1 + \frac{1}{i\not{D} - m} \sum_{a=1}^7 \sum_{j=1}^2 \tau_a \left[ \sigma_j^a + i\gamma_5 \varphi_j^a \right] f_j^a \right] \end{aligned} \quad (92)$$

where  $\sigma_j^a$  and  $\varphi_j^a$  are the scalar and pseudoscalar fields:  $\sum_{a=1}^3 (\sigma_j^a)^2 \equiv a_{0,j}^2 = (a_{0,j}^0)^2 + 2a_{0,j}^+ a_{0,j}^-$ ,  $\sum_{a=4}^7 (\sigma_j^a)^2 \equiv 2K_{0,j}^{*2} = 2(\bar{K}_{0,j}^*)^0 (K_{0,j}^*)^0 + 2(K_{0,j}^+)^+ (K_{0,j}^+)^-$ ,  $\sum_{a=1}^3 (\varphi_j^a)^2 \equiv \pi_j^2 = (\pi_j^0)^2 + 2\pi_j^+ \pi_j^-$ ,  $\sum_{a=4}^7 (\varphi_j^a)^2 \equiv 2K_j^2 = 2\bar{K}_j^0 K_j^0 + 2\bar{K}_j^+ K_j^-$ . As to the coupling constants  $G_{aa}$ , they will be defined later (see Subsec. 5 and (61)).

The free part of Lagrangian (92) has the following form

$$\mathcal{L}^{(2)}(\sigma, \varphi) = \frac{1}{2} \sum_{i,j=1}^2 \sum_{a=1}^7 \left( \sigma_i^a K_{\sigma,ij}^a(P) \sigma_j^a + \varphi_i^a K_{\varphi,ij}^a(P) \varphi_j^a \right) \quad (93)$$

where the coefficients  $K_{\sigma(\varphi),ij}^a(P)$  are given by

$$\begin{aligned} K_{\sigma(\varphi),ij}^a(P) = & -\delta_{ij} \left[ \frac{\delta_{i1}}{G_{aa}^{(\mp)}} + \frac{\delta_{i2}}{G} \right] - \\ & iN_c \text{Tr} \int_{\Lambda_3} \frac{d^4 k}{(2\pi)^4} \frac{1}{\not{k} + \not{P}/2 - m_q^a} r^{\sigma(\varphi)} f_i^a \frac{1}{\not{k} - \not{P}/2 - m_{q'}^a} r^{\sigma(\varphi)} f_j^a, \end{aligned} \quad (94)$$

$$r^\sigma = 1, \quad r^\phi = i\gamma_5, \quad (95)$$

$$m_q^a = m_u \quad (a = 1, \dots, 7); \quad m_{q'}^a = m_u \quad (a = 1, \dots, 3); \quad m_{q'}^a = m_s \quad (a = 4, \dots, 7), \quad (96)$$

with  $m_u$  and  $m_s$  being the constituent quark masses and  $f_j^a$  being defined in (65). Integral (94) is evaluated by expanding in the meson field momentum  $P$ . To order  $P^2$ , one obtains

$$\begin{aligned} K_{\sigma(\varphi),11}^a(P) &= Z_{\sigma(\varphi),1}^a(P^2 - (m_q^a \pm m_{q'}^a)^2 - M_{\sigma^a(\varphi^a),1}^2), \\ K_{\sigma(\varphi),22}^a(P) &= Z_{\sigma(\varphi),2}^a(P^2 - (m_q^a \pm m_{q'}^a)^2 - M_{\sigma^a(\varphi^a),2}^2), \\ K_{\sigma(\varphi),12}^a(P) &= K_{\sigma(\varphi),21}^a(P) = \gamma_{\sigma(\varphi)}^a(P^2 - (m_q^a \pm m_{q'}^a)^2) \end{aligned} \quad (97)$$

where

$$Z_{\sigma,1}^a = 4I_2^a, \quad Z_{\sigma,2}^a = 4I_2^{ffa}, \quad \gamma_\sigma^a = 4I_2^{fa}, \quad (98)$$

$$Z_{\varphi,1}^a = Z Z_{\sigma,1}^a, \quad Z_{\varphi,2}^a = Z_{\sigma,2}^a, \quad \gamma_\varphi^a = Z^{1/2} \gamma_\sigma^a \quad (99)$$

and

$$M_{\sigma^a(\varphi^a),1}^2 = (Z_{\sigma(\varphi),1}^a)^{-1} \left[ \frac{1}{G_{aa}^{(\mp)}} - 4(I_1(m_q^a) + I_1(m_{q'}^a)) \right], \quad (100)$$

$$M_{\sigma^a(\varphi^a),2}^2 = (Z_{\sigma(\varphi),2}^a)^{-1} \left[ \frac{1}{G} - 4(I_1^{ffa}(m_q^a) + I_1^{ffa}(m_{q'}^a)) \right]. \quad (101)$$

The factor  $Z$  here appears due to  $\pi - a_1$ -transitions [8, 10, 28] (see Subsec. 3.1), and the integrals  $I_2^{f..f}$  contain form factors:

$$I_2^{f..fa}(m_q^a, m_{q'}^a) = \frac{-iN_c}{(2\pi)^4} \int_{\Lambda_3} d^4k \frac{f_a(\mathbf{k}) \cdot f_a(\mathbf{k})}{((m_q^a)^2 - k^2)((m_{q'}^a)^2 - k^2)}. \quad (102)$$

After the renormalization of the scalar fields

$$\sigma_i^{ar} = \sqrt{Z_{\sigma,i}^a} \sigma_i^a, \quad \varphi_i^{ar} = \sqrt{Z_{\varphi,i}^a} \varphi_i^a \quad (103)$$

the part of Lagrangian (93) that describes the scalar and pseudoscalar mesons takes the form

$$\begin{aligned} \mathcal{L}_{a_0}^{(2)} &= \frac{1}{2} (P^2 - 4m_u^2 - M_{a_0,1}^2) a_{0,1}^2 + \Gamma_{a_0} (P^2 - 4m_u^2) a_{0,1} a_{0,2} \\ &+ \frac{1}{2} (P^2 - 4m_u^2 - M_{a_0,2}^2) a_{0,2}^2, \end{aligned} \quad (104)$$

$$\begin{aligned} \mathcal{L}_{K_0^*}^{(2)} &= \frac{1}{2} (P^2 - (m_u + m_s)^2 - M_{K_0^*,1}^2) K_{0,1}^{*2} + \Gamma_{K_0^*} (P^2 - (m_u + m_s)^2) K_{0,1}^* K_{0,2}^* \\ &+ \frac{1}{2} (P^2 - (m_u + m_s)^2 - M_{K_0^*,2}^2) K_{0,2}^{*2}, \end{aligned} \quad (105)$$

$$\mathcal{L}_\pi^{(2)} = \frac{1}{2} (P^2 - M_{\pi,1}^2) \pi_1^2 + \Gamma_\pi P^2 \pi_1 \pi_2 + \frac{1}{2} (P^2 - M_{\pi,2}^2) \pi_2^2, \quad (106)$$

$$\mathcal{L}_K^{(2)} = \frac{1}{2} (P^2 - M_{K,1}^2) K_1^2 + \Gamma_K P^2 K_1 K_2 + \frac{1}{2} (P^2 - M_{K,2}^2) K_2^2, \quad (107)$$

where

$$\Gamma_{\sigma^a} = \frac{I_2^{f_a}}{\sqrt{I_2 I_2^{ff_a}}}, \quad \Gamma_{\varphi^a} = Z^{-1/2} \Gamma_{\sigma^a}. \quad (108)$$

After the transformations of the meson fields

$$\begin{aligned} \sigma^a &= \cos(\theta_{\sigma,a} - \theta_{\sigma,a}^0) \sigma_1^{ar} - \cos(\theta_{\sigma,a} + \theta_{\sigma,a}^0) \sigma_2^{ar}, \\ \hat{\sigma}^a &= \sin(\theta_{\sigma,a} - \theta_{\sigma,a}^0) \sigma_1^{ar} - \sin(\theta_{\sigma,a} + \theta_{\sigma,a}^0) \sigma_2^{ar}, \end{aligned} \quad (109)$$

$$\begin{aligned} \varphi^a &= \cos(\theta_{\varphi,a} - \theta_{\varphi,a}^0) \varphi_1^{ar} - \cos(\theta_{\varphi,a} + \theta_{\varphi,a}^0) \varphi_2^{ar}, \\ \hat{\varphi}^a &= \sin(\theta_{\varphi,a} - \theta_{\varphi,a}^0) \varphi_1^{ar} - \sin(\theta_{\varphi,a} + \theta_{\varphi,a}^0) \varphi_2^{ar}. \end{aligned} \quad (110)$$

Lagrangians (104), (105), (106), and (107) assume the diagonal form:

$$L_{a_0}^{(2)} = \frac{1}{2}(P^2 - M_{a_0}^2) a_0^2 + \frac{1}{2}(P^2 - M_{\hat{a}_0}^2) \hat{a}_0^2, \quad (111)$$

$$L_{K_0^*}^{(2)} = \frac{1}{2}(P^2 - M_{K_0^*}^2) K_0^{*2} + \frac{1}{2}(P^2 - M_{\hat{K}_0^*}^2) \hat{K}_0^{*2}. \quad (112)$$

$$L_{\pi}^{(2)} = \frac{1}{2}(P^2 - M_{\pi}^2) \pi^2 + \frac{1}{2}(P^2 - M_{\hat{\pi}}^2) \hat{\pi}^2, \quad (113)$$

$$L_K^{(2)} = \frac{1}{2}(P^2 - M_K^2) K^2 + \frac{1}{2}(P^2 - M_{\hat{K}}^2) \hat{K}^2. \quad (114)$$

Here we have

$$\begin{aligned} M_{(a_0, \hat{a}_0)}^2 &= \frac{1}{2(1 - \Gamma_{a_0}^2)} \left[ M_{a_0,1}^2 + M_{a_0,2}^2 \right. \\ &\quad \left. \pm \sqrt{(M_{a_0,1}^2 - M_{a_0,2}^2)^2 + (2M_{a_0,1}M_{a_0,2}\Gamma_{a_0})^2} \right] + 4m_u^2, \end{aligned} \quad (115)$$

$$\begin{aligned} M_{(K_0^*, \hat{K}_0^*)}^2 &= \frac{1}{2(1 - \Gamma_{K_0^*}^2)} \left[ M_{K_0^*,1}^2 + M_{K_0^*,2}^2 \right. \\ &\quad \left. \pm \sqrt{(M_{K_0^*,1}^2 - M_{K_0^*,2}^2)^2 + (2M_{K_0^*,1}M_{K_0^*,2}\Gamma_{K_0^*})^2} \right] + (m_u + m_s)^2, \end{aligned} \quad (116)$$

$$M_{(\pi, \hat{\pi})}^2 = \frac{1}{2(1 - \Gamma_{\pi}^2)} \left[ M_{\pi,1}^2 + M_{\pi,2}^2 \pm \sqrt{(M_{\pi,1}^2 - M_{\pi,2}^2)^2 + (2M_{\pi,1}M_{\pi,2}\Gamma_{\pi})^2} \right], \quad (117)$$

$$M_{(K, \hat{K})}^2 = \frac{1}{2(1 - \Gamma_K^2)} \left[ M_{K,1}^2 + M_{K,2}^2 \pm \sqrt{(M_{K,1}^2 - M_{K,2}^2)^2 + (2M_{K,1}M_{K,2}\Gamma_K)^2} \right], \quad (118)$$

and

$$\tan 2\bar{\theta}_{\sigma(\varphi),a} = \sqrt{\frac{1}{\Gamma_{\sigma^a(\varphi^a)}^2} - 1} \left[ \frac{M_{\sigma^a(\varphi^a),1}^2 - M_{\sigma^a(\varphi^a),2}^2}{M_{\sigma^a(\varphi^a),1}^2 + M_{\sigma^a(\varphi^a),2}^2} \right], \quad 2\theta_{\sigma(\varphi),a} = 2\bar{\theta}_{\sigma(\varphi),a} + \pi, \quad (119)$$

$$\sin \theta_{\sigma(\varphi),a}^0 = \sqrt{\frac{1 + \Gamma_{\sigma^a(\varphi^a)}}{2}}. \quad (120)$$



Table 1: The mixing coefficients for the ground and first radially excited states of scalar and pseudoscalar isovector and strange mesons. The caret symbol marks the excited states.

	$a_0$	$\hat{a}_0$		$K_0^*$	$\hat{K}_0^*$
$a_{0,1}$	0.87	0.82	$K_{0,1}^*$	0.83	0.89
$a_{0,2}$	0.22	-1.17	$K_{0,2}^*$	0.28	-1.11

	$\pi$	$\hat{\pi}$		$K$	$\hat{K}$
$\pi_1$	1.00	0.54	$K_1$	0.96	0.56
$\pi_2$	0.01	-1.14	$K_2$	0.09	-1.11

The caret symbol stands for the first radial excitations of mesons. Transformations (109) and (110) express the “physical” fields  $\sigma$ ,  $\varphi$ ,  $\hat{\sigma}$ , and  $\hat{\varphi}$  through the “bare” ones  $\sigma_i^{ar}$ ,  $\varphi_i^{ar}$  and for calculations these equations must be inverted. For practical use, we collect the values of coefficients in the inverted equations for the scalar and pseudoscalar fields in Table 1.

For the weak decay constants of pions and kaons we obtain

$$F_\pi = 2m_u \sqrt{ZI_2(m_u)} \cos(\theta_\pi - \theta_\pi^0), \quad (121)$$

$$F_{\pi'} = 2m_u \sqrt{ZI_2(m_u)} \sin(\theta_\pi - \theta_\pi^0), \quad (122)$$

$$F_K = (m_u + m_s) \sqrt{ZI_2(m_u, m_s)} \cos(\theta_K - \theta_K^0), \quad (123)$$

$$F_{K'} = (m_u + m_s) \sqrt{ZI_2(m_u, m_s)} \sin(\theta_K - \theta_K^0). \quad (124)$$

In the chiral limit we have  $\theta_a = \theta_a^0$  and

$$F_\pi = \frac{m_u}{g_\pi}, \quad F_K = \frac{m_u + m_s}{2g_K} \quad (125)$$

$$F_{\pi'} = F_{K'} = 0, \quad g_\pi = (Z_1^\pi)^{-1/2}, \quad g_K = (Z_1^K)^{-1/2}. \quad (126)$$

As one can see from these formulae, in the chiral limit we obtain the Goldberger-Treiman identities for the coupling constants  $g_\pi$  and  $g_K$ . The matrix elements of divergences of the axial currents between meson states and vacuum equal (PCAC relations) are

$$\langle 0 | \partial^\mu A_\mu^a | \varphi \rangle = M_\varphi^2 F_\varphi \delta^{ab}, \quad (127)$$

$$\langle 0 | \partial^\mu A_\mu^a | \varphi' \rangle = M_{\varphi'}^2 F_{\varphi'} \delta^{ab}. \quad (128)$$

These axial currents are conserved in the chiral limit because their divergences equal zero, according to low-energy theorems.

### 3.3 The masses of isoscalar mesons (the ground and excited states)

The free part of the effective Lagrangian for isoscalar scalar and pseudoscalar mesons after bosonization is as follows

$$\begin{aligned}\mathcal{L}_{\text{isosc}}(\sigma, \varphi) = & -\frac{1}{2} \sum_{a,b=8}^9 \left[ \sigma_1^a (G^{(-)})_{ab}^{-1} \sigma_1^b + \varphi_1^a (G^{(+)})_{ab}^{-1} \varphi_1^b \right] \\ & -\frac{1}{2G} \sum_{a=8}^9 \left[ (\sigma_2^a)^2 + (\varphi_2^a)^2 \right] \\ & -i \text{Tr} \ln \left\{ 1 + \frac{1}{i\cancel{D} - m} \sum_{a=8}^9 \sum_{j=1}^2 \tau^a [\sigma_j^a + i\gamma_5 \varphi_j^a] f_j^a \right\},\end{aligned}\quad (129)$$

where  $(G^{(\pm)})^{-1}$  is the inverse of  $G^{(\pm)}$ :

$$\begin{aligned}(G^{(\mp)})_{88}^{-1} &= G_{88}^{(\mp)} / D^{(\mp)}, & (G^{(\mp)})_{89}^{-1} &= (G^{(\mp)})_{98}^{-1} = -G_{89}^{(\mp)} / D^{(\mp)}, \\ (G^{(\mp)})_{99}^{-1} &= G_{99}^{(\mp)} / D^{(\mp)}, & D^{(\mp)} &= G_{88}^{(\mp)} G_{99}^{(\mp)} - (G_{89}^{(\mp)})^2.\end{aligned}\quad (130)$$

From (129), in the one-loop approximation, one obtains the free part of the effective Lagrangian

$$\mathcal{L}^{(2)}(\sigma, \phi) = \frac{1}{2} \sum_{i,j=1}^2 \sum_{a,b=8}^9 \left( \sigma_i^a K_{\sigma,ij}^{[a,b]}(P) \sigma_j^b + \varphi_i^a K_{\phi,ij}^{[a,b]}(P) \varphi_j^b \right). \quad (131)$$

The definition of  $K_{\sigma(\varphi),i}^{[a,b]}$  is given in Appendix A.

After the renormalization of both the scalar and pseudoscalar fields, analogous to (103), we come to the Lagrangian that can be represented in a form slightly different from that of (131). It is convenient to introduce 4-vectors of “bare” fields

$$\Sigma = (\sigma_1^{8r}, \sigma_2^{8r}, \sigma_1^{9r}, \sigma_2^{9r}), \quad \Phi = (\varphi_1^{8r}, \varphi_2^{8r}, \varphi_1^{9r}, \varphi_2^{9r}). \quad (132)$$

Thus, we have

$$\mathcal{L}^{(2)}(\Sigma, \Phi) = \frac{1}{2} \sum_{i,j=1}^4 (\Sigma_i \mathcal{K}_{\Sigma,ij}(P) \Sigma_j + \Phi_i \mathcal{K}_{\Phi,ij}(P) \Phi_j) \quad (133)$$

where we introduced new functions  $\mathcal{K}_{\Sigma(\Phi),ij}(P)$  (see Appendix A). The index  $r$  marks renormalized fields.

Up to this moment we have four pseudoscalar and four scalar meson states which are the octet and nonet singlets. Mesons of the same parity have the same quantum numbers and, therefore, they are expected to be mixed. In our model the mixing is represented by  $4 \times 4$  matrices  $R^{\sigma(\varphi)}$  which transform the “bare” fields  $\sigma_i^{8r}$ ,  $\sigma_i^{9r}$ ,  $\varphi_i^{8r}$  and  $\varphi_i^{9r}$  entering into the 4-vectors  $\Sigma$  and  $\Phi$  into the “physical” ones  $\sigma$ ,  $\hat{\sigma}$ ,  $f_0$ ,  $\hat{f}_0$ ,  $\eta$ ,  $\eta'$ ,  $\hat{\eta}$ , and  $\hat{\eta}'$  represented as components of the vectors  $\Sigma_{\text{ph}}$  and  $\Phi_{\text{ph}}$ :

$$\Sigma_{\text{ph}} = (\sigma, \hat{\sigma}, f_0, \hat{f}_0), \quad \Phi_{\text{ph}} = (\eta, \hat{\eta}, \eta', \hat{\eta}'). \quad (134)$$

Table 2: The mixing coefficients for isoscalar meson states

	$\eta$	$\hat{\eta}$	$\eta'$	$\hat{\eta}'$
$\varphi_1^8$	0.71	0.62	-0.32	0.56
$\varphi_2^8$	0.11	-0.87	-0.48	-0.54
$\varphi_1^9$	0.62	0.19	0.56	-0.67
$\varphi_2^9$	0.06	-0.66	0.30	0.82

	$\sigma$	$\hat{\sigma}$	$f_0$	$\hat{f}_0$
$\sigma_1^8$	-0.98	-0.66	0.10	0.17
$\sigma_2^8$	0.02	1.15	0.26	-0.17
$\sigma_1^9$	0.27	-0.09	0.82	0.71
$\sigma_2^9$	-0.03	-0.21	0.22	-1.08

The transformation  $R^{\sigma(\varphi)}$  is linear and nonorthogonal:

$$\Sigma_{\text{ph}} = R^\sigma \Sigma, \quad \Phi_{\text{ph}} = R^\varphi \Phi. \quad (135)$$

In terms of “physical” fields the free part of the effective Lagrangian is of the conventional form and the coefficients of matrices  $R^{\sigma(\varphi)}$  give the mixing of the  $\bar{u}u$  and  $\bar{s}s$  components, with and without form factors.

Because of complexity of the procedure of diagonalization for the matrices of dimensions greater than 2, there are no such simple formulae as, *e.g.*, (109). Hence, we do not implement it analytically but use numerical methods to obtain matrix elements (see Table 2).

### 3.4 The effective Lagrangian for the ground and excited states of vector mesons

The free part of the effective Lagrangian (66) describing the ground and excited states of vector mesons has the form

$$L^{(2)}(V) = -\frac{1}{2} \sum_{i,j=1}^2 \sum_{a=1}^9 V_i^{a\mu}(P) R_{ij}^{a\mu\nu}(P) V_j^{a\nu}(P), \quad (136)$$

where

$$\begin{aligned} \sum_{a=1}^3 (V_i^{a\mu})^2 &= (\rho_i^{0\mu})^2 + 2\rho_i^{+\mu} \rho_i^{-\mu}, & (V_i^{4\mu})^2 + (V_i^{5\mu})^2 &= 2K_i^{*+\mu} K_i^{*-\mu}, \\ (V_i^{6\mu})^2 + (V_i^{7\mu})^2 &= 2K_i^{*0\mu} K_i^{*0\mu}, & (V_i^{8\mu})^2 &= (\omega_i^\mu)^2, & (V_i^{9\mu})^2 &= (\varphi_i^\mu)^2 \end{aligned} \quad (137)$$

and

$$\begin{aligned} R_{ij}^{a\mu\nu}(P) &= -\frac{\delta_{ij}}{G_V} g^{\mu\nu} \\ &- i N_c \text{tr} \int_{\Lambda_3} \frac{d^4 k}{(2\pi)^4} \left[ \frac{1}{\not{k} + \frac{1}{2}\not{P} - m_q^a} \gamma^\mu f_i^{a,V} \frac{1}{\not{k} - \frac{1}{2}\not{P} - m_{q'}^a} \gamma^\nu f_j^{a,V} \right], \end{aligned} \quad (138)$$

To order  $P^2$ , one obtains

$$\begin{aligned} R_{11}^{a\mu\nu} &= W_1^a [P^2 g^{\mu\nu} - P^\mu P^\nu - g^{\mu\nu} (\bar{M}_1^a)^2], \\ R_{22}^{a\mu\nu} &= W_2^a [P^2 g^{\mu\nu} - P^\mu P^\nu - g^{\mu\nu} (\bar{M}_2^a)^2], \\ R_{12}^{a\mu\nu} &= R_{21}^{\mu\nu a} = \bar{\gamma}^a [P^2 g^{\mu\nu} - P^\mu P^\nu - \frac{3}{2} (m_q^a - m_{q'}^a)^2 g^{\mu\nu}]. \end{aligned} \quad (139)$$

Here

$$W_1^a = \frac{8}{3} I_2^a, \quad W_2^a = \frac{8}{3} I_2^{fa}, \quad \bar{\gamma}^a = \frac{8}{3} I_2^{fa}, \quad (140)$$

$$(\bar{M}_1^a)^2 = (W_1^a G_V)^{-1} + \frac{3}{2} (m_q^a - m_{q'}^a)^2, \quad (141)$$

$$(\bar{M}_2^a)^2 = (W_2^a G_V)^{-1} + \frac{3}{2} (m_q^a - m_{q'}^a)^2. \quad (142)$$

After renormalization of the meson fields

$$V_i^{a\mu} = \sqrt{W_i^a} V_i^{a\mu} \quad (143)$$

we obtain the Lagrangians

$$\begin{aligned} L_\rho^{(2)} &= -\frac{1}{2} \left[ (g^{\mu\nu} P^2 - P^\mu P^\nu - g^{\mu\nu} M_{\rho_1}^2) \rho_1^\mu \rho_1^\nu \right. \\ &\quad \left. + 2\Gamma_\rho (g^{\mu\nu} P^2 - P^\mu P^\nu) \rho_1^\mu \rho_2^\nu + (g^{\mu\nu} P^2 - P^\mu P^\nu - g^{\mu\nu} M_{\rho_2}^2) \rho_2^\mu \rho_2^\nu \right], \end{aligned} \quad (144)$$

$$\begin{aligned} L_\varphi^{(2)} &= -\frac{1}{2} \left[ (g^{\mu\nu} P^2 - P^\mu P^\nu - g^{\mu\nu} M_{\varphi_1}^2) \varphi_1^\mu \varphi_1^\nu \right. \\ &\quad \left. + 2\Gamma_\varphi (g^{\mu\nu} P^2 - P^\mu P^\nu) \varphi_1^\mu \varphi_2^\nu + (g^{\mu\nu} P^2 - P^\mu P^\nu - g^{\mu\nu} M_{\varphi_2}^2) \varphi_2^\mu \varphi_2^\nu \right], \end{aligned} \quad (145)$$

$$\begin{aligned} L_{K^*}^{(2)} &= -\frac{1}{2} \left[ (g^{\mu\nu} P^2 - P^\mu P^\nu - g^{\mu\nu} \left( \frac{3}{2} (m_q^a - m_{q'}^a)^2 + M_{K_1^*}^2 \right)) K_1^{*\mu} K_1^{*\nu} \right. \\ &\quad + 2\Gamma_{K^*} (g^{\mu\nu} P^2 - P^\mu P^\nu - g^{\mu\nu} \frac{3}{2} (m_q^a - m_{q'}^a)^2) K_1^{*\mu} K_2^{*\nu} \\ &\quad \left. + (g^{\mu\nu} P^2 - P^\mu P^\nu - g^{\mu\nu} \left( \frac{3}{2} (m_q^a - m_{q'}^a)^2 + M_{K_2^*}^2 \right)) K_2^{*\mu} K_2^{*\nu} \right]. \end{aligned} \quad (146)$$

Here

$$\begin{aligned} M_{\rho_1}^2 &= \frac{3}{8G_V I_2(m_u, m_u)}, \quad M_{K^*1}^2 = \frac{3}{8G_V I_2(m_u, m_s)}, \\ M_{\varphi_1}^2 &= \frac{3}{8G_V I_2(m_s, m_s)}, \quad M_{\rho_2}^2 = \frac{3}{8G_V I_2^{ff}(m_u, m_u)}, \\ M_{K^*2}^2 &= \frac{3}{8G_V I_2^{ff}(m_u, m_s)}, \quad M_{\phi_2}^2 = \frac{3}{8G_V I_2^{ff}(m_s, m_s)}, \end{aligned} \quad (147)$$

$$\Gamma_{a_i, j} = \frac{I_2^{fa}(m_i, m_j)}{\sqrt{I_2^a(m_i, m_j) I_2^{ffa}(m_i, m_j)}}. \quad (148)$$

After transformations of the vector meson fields, similar to Eqs. (109) for the pseudoscalar mesons, Lagrangians (144), (145), (146) take the diagonal form

$$L_{V^a, \bar{V}^a}^{(2)} = -\frac{1}{2} \left[ (g^{\mu\nu} P^2 - P^\mu P^\nu - M_{V^a}^2) V^{a\mu} V^{a\nu} + (g^{\mu\nu} P^2 - P^\mu P^\nu - M_{\bar{V}^a}^2) \bar{V}^{a\mu} \bar{V}^{a\nu} \right], \quad (149)$$

where  $V^{a\mu}$  and  $\bar{V}^{a\mu}$  are physical ground and excited states of vector mesons

$$\begin{aligned} M_{\rho, \bar{\rho}}^2 &= \frac{1}{2(1 - \Gamma_\rho^2)} \left[ M_{\rho_1}^2 + M_{\rho_2}^2 \mp \sqrt{(M_{\rho_1}^2 - M_{\rho_2}^2)^2 + (2M_{\rho_1} M_{\rho_2} \Gamma_\rho)^2} \right] \\ &= M_{\omega, \bar{\omega}}^2, \end{aligned} \quad (150)$$

$$M_{\varphi, \bar{\varphi}}^2 = \frac{1}{2(1 - \Gamma_\phi^2)} \left[ M_{\varphi_1}^2 + M_{\varphi_2}^2 \mp \sqrt{(M_{\varphi_1}^2 - M_{\varphi_2}^2)^2 + (2M_{\varphi_1} M_{\varphi_2} \Gamma_\varphi)^2} \right], \quad (151)$$

$$\begin{aligned} M_{K^*, \bar{K}^*}^2 &= \frac{1}{2(1 - \Gamma_{K^*}^2)} \left[ M_{K_1^*}^2 + M_{K_2^*}^2 + 3\Delta^2(1 - \Gamma_{K^*}^2) \right. \\ &\quad \left. \mp \sqrt{(M_{K_1^*}^2 - M_{K_2^*}^2)^2 + (2M_{K_1^*} M_{K_2^*} \Gamma_{K^*})^2} \right]. \end{aligned} \quad (152)$$

### 3.5 Numerical estimates.

In our model we have six basic parameters (see Subsection 3.1): the masses of the constituent  $u(d)$  and  $s$  quarks,  $m_u = m_d$  and  $m_s$ , the cut-off parameter  $\Lambda_3$ , two four-quark coupling constants (one for the scalar and pseudoscalar channels,  $G$ , and the other for the vector and axial-vector channels,  $G_V$ ) and the 't Hooft coupling constant  $K$ . We fixed these parameters with the help of input parameters: the pion decay constant  $F_\pi = 93$  MeV, the  $\rho$ -meson decay constant  $g_\rho = 6.14$  (decay  $\rho \rightarrow 2\pi$ )<sup>5</sup>, the masses of pion, kaon,  $\rho$ -meson, and the mass difference of  $\eta$  and  $\eta'$  mesons. Using mass formulae given in previous subsections of this Section, we obtain numerical estimates of these parameters:

$$\begin{aligned} m_u &= 280 \text{ MeV}, \quad m_s = 405 \text{ MeV}, \quad \Lambda_3 = 1.03 \text{ GeV}, \\ G &= 3.14 \text{ GeV}^{-2}, \quad G_V = 12 \text{ GeV}^{-2}, \quad K = 6.1 \text{ GeV}^{-5}. \end{aligned} \quad (153)$$

When excited meson states are introduced, a set of additional parameters related to the form factors appears in our model: the slope parameters  $d_{qq}$  and the external parameters  $c_{qq}^U$ . The slope parameters  $d_{qq}$  are fixed by special conditions (see Eqs.(73), (74), (75)) from which we obtain:  $d_{uu} = -1.78 \text{ GeV}^{-2}$ ,  $d_{us} = -1.76 \text{ GeV}^{-2}$ ,  $d_{ss} = -1.73 \text{ GeV}^{-2}$ . As it was mentioned earlier, we assume here that  $d_{uu}$ ,  $d_{us}$ , and  $d_{ss}$  do not depend on parity and spin of mesons.

The parameters  $c_{qq}^{\sigma^a(\varphi^a)}$  are fitted by masses of excited pseudoscalar and vector mesons,  $c_{uu}^{\pi, a_0} = 1.44$ ,  $c_{uu}^{\eta, \eta', \sigma, f_0} = 1.5$ ,  $c_{us}^{K, K^*} = 1.59$ ,  $c_{ss}^{\eta, \eta', \sigma, f_0} = 1.66$ ,  $c_{uu}^\rho = 1.33$ ,  $c_{us}^{K^*} = 1.6$ ,  $c_{ss}^\varphi = 1.41$ . These parameters characterize how stronger the quark currents with form factors attract

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<sup>5</sup>Here, we used the relation  $g_\rho = \sqrt{6}g_\sigma$  together with the Goldberger–Treiman relation  $g_\pi = m/F_\pi = Z^{-1/2}g_\sigma$  to fix the parameters  $m_u$  and  $\Lambda_3$ .

Table 3: The model masses of mesons, MeV

	<i>GR</i>	<i>EXC</i>	<i>GR(Exp.)</i> [1]	<i>EXC(Exp.)</i> [1]
$M_\sigma$	530	1330	400 – 1200	1200 – 1500
$M_{f_0}$	1070	1600	$980 \pm 10$	$1712 \pm 5$
$M_{a_0}$	830	1500	$983.4 \pm 0.9$	$1474 \pm 19$
$M_{K_0^*}$	960	1500	$905 \pm 50$ [5]	$1429 \pm 12$
$M_\pi$	140	1300	$139.56995 \pm 0.00035$	$1300 \pm 100$
$M_K$	490	1300	$497.672 \pm 0.031$	1460(?)
$M_\eta$	520	1280	$547.30 \pm 0.12$	$1297.8 \pm 2.8$
$M_{\eta'}$	910	1470	$957.78 \pm 0.14$	1440 – 1470
$M_\rho$	770	1470	$770.0 \pm 0.8$	$1465 \pm 25$
$M_\varphi$	1019	1682	$1019.413 \pm 0.008$	$1680 \pm 20$
$M_{K^*}$	887	1479	$891.59 \pm 0.24$	$1412 \pm 12$

to each other than those without form factors. We use the same parameters for the scalar and pseudoscalar mesons (global chiral symmetry). This allows us to predict the masses of ground and excited states of scalar mesons. The result is represented in Table 3 together with experimental values.

We also calculate the angles  $\theta_a$  and  $\theta_a^0$ :

$$\begin{aligned}
\theta_\pi &= 59.48^\circ & \theta_\pi^0 &= 59.12^\circ, & \theta_K &= 60.2^\circ, \\
\theta_K^0 &= 57.13^\circ, & \theta_\rho &= 81.8^\circ, & \theta_\rho^0 &= 81.5^\circ \\
\theta_{K^*} &= 84.7^\circ, & \theta_{K^*}^0 &= 59.14^\circ, & \theta_\varphi &= 68.4^\circ, \\
\theta_\varphi^0 &= 57.13^\circ, & \theta_{a_0} &= 72.0^\circ, & \theta_{a_0}^0 &= 61.5^\circ, \\
\theta_{K_0^*} &= 74.0^\circ & \theta_{K_0^*}^0 &= 60.0^\circ.
\end{aligned} \tag{154}$$

We consider it expedient to give the values of angles because they will be used in the next Section when the calculation of strong decays of the ground and first radially excited states of the  $\pi$  and  $\rho$  meson will be treated in detail. However, the mixing coefficients for  $\pi$ ,  $K$ ,  $a_0$ , and  $K_0^*$  defined by these angles have been displayed in Table 1. The mixing coefficients for  $\eta$ ,  $\eta'$ ,  $\sigma$ , and  $f_0$  are given in Table 2.

Having fixed all parameters in our model, we can predict the masses of  $\eta$ ,  $\eta'$ ,  $K_0^*$ , and  $\varphi$  mesons and all masses of the ground and first radially excited scalar meson states. We also calculate the weak decay constants for the pion and kaon (both for the ground and excited states):

$$F_\pi = 93 \text{ MeV}, \quad F_{\pi'} = 0.57 \text{ MeV}, \tag{155}$$

$$F_K = 1.16 F_\pi = 108 \text{ MeV}, \quad F_{K'} = 3.3 \text{ MeV}. \tag{156}$$

Moreover, now we are able to estimate all strong coupling constants for the mesons considered in this paper. In the next Section we calculate some of these constants that define the strong

decay processes of ground states and first radial excitations of the scalar, pseudoscalar, and vector meson nonets.

## 4. Strong decays of mesons.

### 4.1 Decays $\rho \rightarrow 2\pi, \pi' \rightarrow \rho\pi, \pi' \rightarrow \sigma\pi, \rho' \rightarrow 2\pi, \rho' \rightarrow \omega\pi$ and $\omega' \rightarrow \rho\pi$ .

In this section we calculate the widths of main decays of scalar, pseudoscalar, and vector meson nonets (for Subsec. 4.1 see refs.[33], for Subsec. 4.2 ref.[34], for Subsecs. 4.3 and 4.4 see ref. [30]) through triangle quark diagrams. When calculating these diagrams, we keep the least possible dependence on external momenta: squared for the anomaly type graphs and linear for nother types. We omit the higher order momentum dependence.

We start with the decay  $\rho \rightarrow 2\pi$ . The amplitude describing this decay has the form

$$T_{\rho \rightarrow 2\pi} = i \frac{g_\rho}{2} \epsilon_{ijk} (p_j - p_k)^\nu \rho_\nu^i \pi^j \pi^k, \quad (157)$$

where  $p_{j,k}$  are pion momenta and  $\epsilon_{ijk}$  is antisymmetric tensor. Using the value  $\alpha_\rho = g_\rho^2/(4\pi) \approx 3$  ( $g_\rho \approx 6.1$ ) of refs. [8, 9, 10, 11] we obtain for the decay width

$$\Gamma_{\rho \rightarrow 2\pi} = \frac{\alpha_\rho}{12 M_\rho^2} (M_\rho^2 - 4 M_\pi^2)^{3/2} \approx 151.5 \text{ MeV}. \quad (158)$$

The experimental value is [1]

$$\Gamma_{\rho \rightarrow 2\pi} = 150.7 \pm 1.2 \text{ MeV}. \quad (159)$$

Now let us calculate this amplitude in our model with the excited states of mesons. To this end, we rewrite the amplitude  $T_{\rho \rightarrow 2\pi}$  in the form

$$T_{\rho \rightarrow 2\pi} = i c_{\rho \rightarrow 2\pi} \epsilon_{ijk} (p_j - p_k)^\nu \rho_\nu^i \pi^j \pi^k, \quad (160)$$

and calculate the factor  $c_{\rho \rightarrow 2\pi}$  in the new model. Using Eqs. (103), (110) and (143) we can find the following expressions for meson fields  $\pi_i$  and  $\rho_i$  from the Lagrangian (66) expressed in terms of the physical states  $\pi, \pi'$  and  $\rho, \rho'$

$$\begin{aligned} \pi_1 &= \frac{\sin(\theta_\pi + \theta_\pi^0)\pi - \cos(\theta_\pi + \theta_\pi^0)\pi'}{\sqrt{Z_1} \sin 2\theta_\pi^0}, \\ \pi_2 &= \frac{\sin(\theta_\pi - \theta_\pi^0)\pi - \cos(\theta_\pi - \theta_\pi^0)\pi'}{\sqrt{Z_2} \sin 2\theta_\pi^0}, \end{aligned} \quad (161)$$

$$\begin{aligned} \rho_1 &= \frac{\sin(\theta_\rho + \theta_\rho^0)\rho - \cos(\theta_\rho + \theta_\rho^0)\rho'}{\sin 2\theta_\rho^0 \sqrt{8/3} I_2}, \\ \rho_2 &= \frac{\sin(\theta_\rho - \theta_\rho^0)\rho - \cos(\theta_\rho - \theta_\rho^0)\rho'}{\sin 2\theta_\rho^0 \sqrt{8/3} I_{2,\rho}^{ff}}, \end{aligned} \quad (162)$$

or, using the values  $I_2 = 0.04$ ,  $I_{2,\rho}^{ff} = 0.0244$ , and  $\theta_\pi$ ,  $\theta_\pi^0$ ,  $\theta_\rho$  and  $\theta_\rho^0$  from Eqs. (154), we obtain <sup>6</sup>

$$\begin{aligned}\pi_1 &= \frac{0.878\pi + 0.48\pi'}{0.88\sqrt{Z_1}}, \quad \pi_2 = \frac{0.0061\pi - \pi'}{0.88\sqrt{Z_2}}, \\ \rho_1 &= (0.744\rho + 0.931\rho') g_\rho/2, \quad \rho_2 = (0.48\rho - 1.445\rho') g_\rho/2.\end{aligned}\quad (163)$$

The decay  $\rho \rightarrow 2\pi$  is described by the quark triangle diagrams with the vertices  $\rho_1(\pi_1^2 + 2\pi_1\pi_2 + \pi_2^2)$  and  $\rho_2(\pi_1^2 + 2\pi_1\pi_2 + \pi_2^2)$  (see Fig. 3). Using Eqs. (161), (162) and (163), we arrive at the factor <sup>7</sup>  $c_{\rho \rightarrow 2\pi}$

$$c_{\rho \rightarrow 2\pi} = c_{\rho_1 \rightarrow 2\pi} + c_{\rho_2 \rightarrow 2\pi} = 0.975 g_\rho/2, \quad (164)$$

$$\begin{aligned}c_{\rho_1 \rightarrow 2\pi} &= \frac{\sin(\theta_\rho + \theta_\rho^0)}{\sin^2 2\theta_\pi^0 \sin 2\theta_\rho^0 \sqrt{8/3} I_2} [(\sin(\theta_\pi + \theta_\pi^0))^2 + 2 \sin(\theta_\pi + \theta_\pi^0) \sin(\theta_\pi - \theta_\pi^0) \Gamma_\pi \\ &+ (\sin(\theta_\pi - \theta_\pi^0))^2 = \sin^2 2\theta_\pi^0] = \frac{\sin(\theta_\rho + \theta_\rho^0)}{\sin 2\theta_\rho^0 \sqrt{8/3} I_2} = 0.745 g_\rho/2, \\ c_{\rho_2 \rightarrow 2\pi} &= \frac{\sin(\theta_\rho - \theta_\rho^0)}{\sin^2 2\theta_\pi^0 \sin 2\theta_\rho^0 \sqrt{8/3} I_{2,\rho}^{ff}} [(\sin(\theta_\pi + \theta_\pi^0))^2 \frac{I_2^f}{I_2} \\ &+ 2 \sin(\theta_\pi + \theta_\pi^0) \sin(\theta_\pi - \theta_\pi^0) \frac{I_2^{ff}}{\sqrt{I_2 I_2^{ff}}} + (\sin(\theta_\pi - \theta_\pi^0))^2 \frac{I_2^{fff}}{I_2^{ff}}] = 0.227 g_\rho/2.\end{aligned}\quad (165)$$

Here we used the values  $I_2^f = 0.0185$ ,  $I_2^{ff} = 0.0289$ ,  $I_2^{fff} = 0.0224$  and the equation  $\Gamma_\pi = -\cos 2\theta_\pi^0$  (it can easily be derived from Eq. (120)). Then the decay width  $\rho \rightarrow 2\pi$  is equal to

$$\Gamma_{\rho \rightarrow 2\pi} \approx 149 \text{ MeV}.\quad (166)$$

In the limit  $f = 0$  ( $\theta_\pi = \theta_\pi^0, \theta_\rho = \theta_\rho^0$ ) from Eqs. (165) one finds

$$c_{\rho \rightarrow 2\pi} = c_{\rho_1 \rightarrow 2\pi} = g_\rho/2, \quad c_{\rho_2 \rightarrow 2\pi} = 0.\quad (167)$$

Now let us consider the decay  $\pi' \rightarrow \rho\pi$ . The amplitude of this decay is of the form

$$T_{\pi' \rightarrow \rho\pi}^\nu = i c_{\pi' \rightarrow \rho\pi} \epsilon_{ijk} (p_j + p_k)^\nu \rho_\nu^i \pi^j \pi^k,\quad (168)$$

where

$$c_{\pi' \rightarrow \rho\pi} = c_{\pi' \rightarrow \rho_1\pi} + c_{\pi' \rightarrow \rho_2\pi}.\quad (169)$$

Then for  $c_{\pi' \rightarrow \rho_1\pi}$  we obtain

$$\begin{aligned}c_{\pi' \rightarrow \rho_1\pi} &= \frac{2}{(\sin 2\theta_\pi^0)^2} [-\sin(\theta_\pi + \theta_\pi^0) \cos(\theta_\pi + \theta_\pi^0) - \sin 2\theta_\pi \Gamma_\pi - \sin(\theta_\pi - \theta_\pi^0) \cos(\theta_\pi - \theta_\pi^0) \\ &= -\sin 2\theta_\pi \cos 2\theta_\pi^0 + \sin 2\theta_\pi \cos 2\theta_\pi^0 = 0] \frac{\sin(\theta_\rho + \theta_\rho^0)}{\sin 2\theta_\rho^0} g_\rho/2 = 0,\end{aligned}\quad (170)$$

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<sup>6</sup>Analogous formulae are obtained for the  $\omega$ -meson.

<sup>7</sup>Taking account of the  $\pi \rightarrow a_1$  transitions on external pion lines, we obtain additional factors  $Z$  ( $\bar{Z}$ ) in the numerators of our triangle diagrams which cancel corresponding factors in  $Z_i$  (see Eqs. (22), (161) and ref. [10]). Therefore, in future we shall ignore the factors  $Z$  ( $\bar{Z}$ ) in  $Z_i$ .



$$c_{\pi' \rightarrow \rho_2 \pi} = \frac{2}{(\sin 2\theta_\pi^0)^2} \left[ -\sin(\theta_\pi + \theta_\pi^0) \cos(\theta_\pi + \theta_\pi^0) \frac{I_2^f}{I_2} - \sin 2\theta_\pi \frac{I_2^{ff}}{\sqrt{I_2 I_2^{ff}}} \right. \\ \left. - \sin(\theta_\pi - \theta_\pi^0) \cos(\theta_\pi - \theta_\pi^0) \frac{I_2^{fff}}{I_2^{ff}} \right] \frac{\sin(\theta_\rho - \theta_\rho^0)}{\sin 2\theta_\rho^0} \sqrt{\frac{I_2}{I_2^{ff}}} g_\rho/2 = -0.573 g_\rho/2. \quad (171)$$

For the decay width  $\pi' \rightarrow \rho\pi$  we get

$$\Gamma_{\pi' \rightarrow \rho\pi} = \frac{c_{\pi' \rightarrow \rho\pi}^2}{4\pi M_{\pi'}^3 M_\rho^2} \Phi(M_{\pi'}, M_\rho, M_\pi)^3 \approx 220 \text{ MeV} \quad (172)$$

where

$$\Phi(M_1, M_2, M_3) = \sqrt{M_1^4 + M_2^4 + M_3^4 - 2(M_1^2 M_2^2 + M_1^2 M_3^2 + M_2^2 M_3^2)}. \quad (173)$$

The decay  $\pi' \rightarrow \sigma\pi$  is calculated in a similar way as  $\hat{\eta} \rightarrow a_0\pi$  (see Subsec. 4.4). Here, we need the mixing coefficients for the scalar meson given in Table 2. We omit details and obtain

$$\Gamma_{\pi' \rightarrow \sigma\pi} \approx 80 \text{ MeV}, \quad (174)$$

therefore, the total width is estimated as

$$\Gamma_\pi^{\text{tot}} \approx 300 \text{ MeV}, \quad (175)$$

This value is in agreement with the experimental data [1]

$$\Gamma_{\pi'}^{\text{tot}} = 200 - 600 \text{ MeV}. \quad (176)$$

For the decay  $\rho' \rightarrow 2\pi$  we arrive in our model at the result

$$\Gamma_{\rho' \rightarrow 2\pi} \approx 22 \text{ MeV}. \quad (177)$$

Most of our results are in agreement with results of the relativized potential quark model with the  $3P_0$ -mechanism of meson decays [2].

To conclude this subsection, we calculate the decay widths of processes  $\rho' \rightarrow \omega\pi$  and  $\omega' \rightarrow \rho\pi$ . These decays go through anomalous triangle quark loop diagrams. The amplitude of the decay  $\rho' \rightarrow \omega\pi$  takes the form

$$T_{\rho' \rightarrow \omega\pi}^{\mu\nu} = \frac{3\alpha_\rho c_{\rho' \rightarrow \omega\pi}}{2\pi F_\pi} \epsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma, \quad (178)$$

where  $q$  and  $p$  are momenta of the  $\omega$  and  $\rho'$  meson, respectively. The factor  $c_{\rho' \rightarrow \omega\pi}$  is similar to the factors  $c_{\rho \rightarrow 2\pi}$  and  $c_{\pi' \rightarrow \rho\pi}$  in the previous equations and arises from the four triangle quark diagrams with vertices  $\pi_1(\rho_1\omega_1 + \rho_2\omega_1 + \rho_1\omega_2 + \rho_2\omega_2)$ <sup>8</sup>. Using the estimate

$$c_{\rho' \rightarrow \omega\pi} \approx -0.3, \quad (179)$$

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<sup>8</sup>We neglect the diagrams with vertices  $\pi_2$ , because their contribution to the ground state of the pion is very small (see Eq.(163)).

we obtain for the decay width

$$\Gamma_{\rho' \rightarrow \omega\pi} = \frac{3}{2\pi M_{\rho'}^3} \left( \frac{\alpha_\rho}{8} \frac{c_{\rho' \rightarrow \omega\pi}}{\pi F_\pi} \right)^2 \Phi(M_{\rho'}, M_\omega, M_\pi)^3 \approx 75 \text{ MeV}. \quad (180)$$

For the decay  $\omega' \rightarrow \rho\pi$  we have the relation

$$\Gamma_{\omega' \rightarrow \rho\pi} \approx 3 \Gamma_{\rho' \rightarrow \omega\pi} \quad (181)$$

leading to the estimate

$$\Gamma_{\omega' \rightarrow \rho\pi} \approx 225 \text{ MeV}. \quad (182)$$

The experimental values are [35]

$$\Gamma_{\rho' \rightarrow \omega\pi}^{exp} = 0.21 \Gamma_{\rho'}^{tot} = 65.1 \pm 12.6 \text{ MeV} \quad (183)$$

and [1]

$$\Gamma_{\omega' \rightarrow \rho\pi}^{exp} = 174 \pm 60 \text{ MeV}. \quad (184)$$

Finally, let us quote the ratio of the decay widths  $\rho' \rightarrow \omega\pi$  and  $\rho' \rightarrow 2\pi$

$$\frac{\Gamma_{\rho' \rightarrow 2\pi}}{\Gamma_{\rho' \rightarrow \omega\pi}} \approx 0.3, \quad (185)$$

which is to be compared with the experimental value 0.32 (see [35]).

Thus, we can see that all our estimates are in satisfactory agreement with experimental data.

Our calculations have shown that the main decay of the  $\rho$ -meson,  $\rho \rightarrow 2\pi$ , changes very little after including the excited meson states into the NJL model. The main part of this decay (75%) comes from the  $\rho$ -vertex without the form factor, whereas the remaining 25% of the decay are due to the  $\rho$ -vertex with the form factor. As a result, the new coupling constant  $g_\rho$  turns out to be very close to the former value.

For the decay  $\pi' \rightarrow \rho\pi$  we meet an opposite situation. Here the channel connected with the  $\rho$ -vertex without the form factor is closed because the states  $\pi$  and  $\pi'$  are orthogonal to each other, and the total decay width of  $\pi' \rightarrow \rho\pi$  is defined by the channel going through the  $\rho$ -vertex with the form factor. As a result, we obtain the quoted value that satisfies experimental data [1]. The decay  $\pi' \rightarrow \sigma\pi$  gives a noticeable correction to the total decay width of  $\pi'$ . These results disagree with the results obtained in the relativized version of the  $3P_1$  potential model [2] in the subject of the  $\pi' \rightarrow \sigma\pi$  decay mode.

For the decay  $\rho' \rightarrow 2\pi$  we obtain strong compensation of the contributions from the two channels, related to  $\rho$ -vertices with and without form factors, and the corresponding decay width is equal to 22 MeV. This value is very close to the result of ref.[2].

It should be emphasized that the decays  $\rho' \rightarrow \omega\pi$  and  $\omega' \rightarrow \rho\pi$  belonging to a different class of quark loop diagrams (“anomaly diagrams”) are also satisfactorily described by our model.

## 4.2 The decays of strange mesons (vectors and pseudoscalars).

In the framework of our model, the decay modes of excited mesons are represented by triangle diagrams with form factors. A total set of diagrams similar to those in Fig. 3 can be represented as one graph: a triangle with shaded angles (see Fig. 4). Every vertex in such diagrams is momentum-dependent and includes form factors defined in Subsection 3.1. For the strange vector and pseudoscalar mesons being decaying, each black shaded vertex with a pseudoscalar meson is implied to contain the following linear combination for the ground state:

$$\bar{f}_a = \frac{1}{\sin 2\theta_a^0} \left[ \frac{\sin(\theta_a + \theta_a^0)}{\sqrt{Z_1^a}} + \frac{\sin(\theta_a - \theta_a^0)}{\sqrt{Z_2^a}} f_a \right], \quad (186)$$

and for an excited state,

$$\bar{f}'_a = \frac{-1}{\sin 2\theta_a^0} \left[ \frac{\cos(\theta_a + \theta_a^0)}{\sqrt{Z_1^a}} + \frac{\cos(\theta_a - \theta_a^0)}{\sqrt{Z_2^a}} f_a \right], \quad (187)$$

where  $\theta_a$  and  $\theta_a^0$  are the angles defined in subsection 3.5 (see Eqs. (119), (120), and (154)) and  $f_a$  is one of the form factors defined in subsection 3.1 (see Eq. (65)). For vector meson vertices, we have the same linear combinations except that  $Z_i^a$  are to be replaced by  $W_i^a$  (140), and the related angles and form factor parameters must be chosen.

Now we can calculate the decay widths of excited mesons. Let us start with the process  $K^{*'} \rightarrow K^* \pi$ . The corresponding amplitude,  $T_{K^{*'} \rightarrow K^* \pi}^{\mu\nu}$ , has the form

$$T_{K^{*'} \rightarrow K^* \pi}^{\mu\nu} = g_{K^{*'} \rightarrow K^* \pi} \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta \quad (188)$$

where  $p$  and  $q$  are momenta of the  $K^{*'}$ - and  $K^*$ -mesons, respectively, and  $g_{K^{*'} \rightarrow K^* \pi}$  is the (dimensional) coupling constant that follows from the combination of one-loop integrals

$$g_{K^{*'} \rightarrow K^* \pi} = \frac{8m_s}{m_u^2 - m_s^2} \left( \mathcal{J}_{2,0}[\bar{f}'_{K^*} \bar{f}_{K^*} \bar{f}_\pi] - \mathcal{J}_{1,1}[\bar{f}'_{K^*} \bar{f}_{K^*} \bar{f}_\pi] \right). \quad (189)$$

In Eq.(189) we introduced a functional defined on functions  $\mathbf{f}(\mathbf{k})$  in the momentum representation:

$$\mathcal{J}_{n,m}[\mathbf{f}] = -i \frac{N_c}{(2\pi)^4} \int_{\Lambda^3} \frac{\mathbf{f}(\mathbf{k}) d^4 k}{(m_u - k^2)^n (m_s - k^2)^m}. \quad (190)$$

This is an alternative to integrals  $I_2^{f \dots f}$  which we thought better to introduce for a growing number of “physical” form factors.

We omit the intermediate calculation here. For the decay constant  $g_{K^{*'} \rightarrow K^* \pi}$  we find

$$g_{K^{*'} \rightarrow K^* \pi} \approx 4 \text{ GeV}^{-1} \quad (191)$$

and the decay width is as follows:

$$\Gamma_{K^{*'} \rightarrow K^* \pi} = \frac{g_{K^{*'} \rightarrow K^* \pi}^2}{32\pi M_{K^{*'}}^3} \Phi(M_{K^{*'}}, M_{K^*}, M_\pi)^3 \approx 90 \text{ MeV}. \quad (192)$$

The lower limit for this value coming from experiment is  $\sim 91 \pm 9 \text{ MeV}$  [1].

A similar calculation has to be performed for the rest of the  $K^{*'}$  decay modes under consideration. The coupling constant  $g_{K^{*'} \rightarrow K\rho}$  is derived in the same way as in (189), with the only difference that  $\bar{f}_\pi$  and  $\bar{f}_{K^*}$  are to be replaced by  $\bar{f}_\rho$  and  $\bar{f}_K$ . The corresponding amplitude,  $T_{K^{*'} \rightarrow K\rho}^{\mu\nu}$ , takes the form

$$T_{K^{*'} \rightarrow K\rho}^{\mu\nu} = g_{K^{*'} \rightarrow K\rho} \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta, \quad (193)$$

where  $p$  and  $q$  are momenta of  $K^{*'}$ - and  $K$ -mesons, respectively, and

$$g_{K^{*'} \rightarrow K\rho} = \frac{8m_s}{m_u^2 - m_s^2} \left( \mathcal{J}_{2,0}[\bar{f}'_{K^*} \bar{f}_K \bar{f}_\rho] - \mathcal{J}_{1,1}[\bar{f}'_{K^*} \bar{f}_K \bar{f}_\rho] \right). \quad (194)$$

The corresponding decay width is

$$\Gamma_{K^{*'} \rightarrow K\rho} = \frac{g_{K^{*'} \rightarrow K\rho}^2}{32\pi M_{K^{*'}}^3} \Phi(M_{K^{*'}}, M_K, M_\rho)^3. \quad (195)$$

For the parameters given in Subsection 3.5 one has

$$g_{K^{*'} \rightarrow K\rho} \approx 3 \text{ GeV}^{-1}, \quad \Gamma_{K^{*'} \rightarrow K\rho} \approx 20 \text{ MeV}. \quad (196)$$

From experiment, the upper limit for this process is  $\Gamma_{K^{*'} \rightarrow K\rho}^{exp} < 16 \pm 1.5 \text{ MeV}$ .

The process  $K^{*'} \rightarrow K\pi$  is described by the amplitude

$$T_{K^{*'} \rightarrow K\pi}^\mu = i \frac{g_{K^{*'} \rightarrow K\pi}}{2} (q - p)^\mu, \quad (197)$$

where  $p$  and  $q$  are momenta of  $\pi$  and  $K$ . The coupling constant  $g_{K^{*'} \rightarrow K\pi}$  is obtained by calculating the one-loop integral

$$g_{K^{*'} \rightarrow K\pi} = 4\mathcal{J}_{1,1}[\bar{f}'_{K^*} \bar{f}_K \bar{f}_\pi] \approx 2 \quad (198)$$

and the decay width is

$$\Gamma_{K^{*'} \rightarrow K\pi} = \frac{g_{K^{*'} \rightarrow K\pi}^2}{64\pi M_{K^{*'}}^5} \Phi(M_{K^{*'}}, M_K, M_\pi)^3 \approx 20 \text{ MeV}. \quad (199)$$

The experimental value is  $15 \pm 5 \text{ MeV}$  [1].

The mesons with hidden strangeness ( $\varphi'$ ) are treated in the same way as  $K^{*'}$ . We consider two decay modes:  $\varphi' \rightarrow KK^*$  and  $\varphi' \rightarrow \bar{K}K$ . Their amplitudes are

$$T_{\varphi' \rightarrow KK^*}^{\mu\nu} = g_{\varphi' \rightarrow KK^*} \epsilon^{\mu\nu\alpha\beta} p_\alpha q_\beta, \quad (200)$$

$$T_{\varphi' \rightarrow \bar{K}K}^\mu = i g_{\varphi' \rightarrow \bar{K}K} (p - q)^\mu. \quad (201)$$

Here,  $p$  and  $q$  are momenta of the  $K$ - and  $K^*$ -mesons. The related coupling constants are

$$g_{\varphi' \rightarrow KK^*} = \frac{8m_u}{m_s^2 - m_u^2} \left( \mathcal{J}_{0,2}[\bar{f}'_\varphi \bar{f}_{K^*} \bar{f}_K] - \mathcal{J}_{1,1}[\bar{f}'_\varphi \bar{f}_{K^*} \bar{f}_K] \right), \quad (202)$$

$$g_{\varphi' \rightarrow \bar{K}K} = 4\mathcal{J}_{0,2}[\bar{f}'_\varphi \bar{f}_K \bar{f}_K]. \quad (203)$$

Thus, the decay widths are estimated as

$$\Gamma_{\varphi' \rightarrow KK^*} \approx 90 \text{ MeV}, \quad \Gamma_{\varphi' \rightarrow \bar{K}K} \approx 10 \text{ MeV}. \quad (204)$$

Unfortunately, there are no reliable experimental data on the partial decay widths for  $\varphi' \rightarrow KK^*$  and  $\varphi' \rightarrow \bar{K}K$  except the total width of  $\varphi'$  being estimated as  $150 \pm 50 \text{ MeV}$  [1]. However, the dominance of the process  $\varphi' \rightarrow KK^*$  is observed is in agreement with our result.

Following the scheme outlined in the previous calculations, we first estimate the  $K' \rightarrow K^*\pi$  and  $K' \rightarrow K\rho$  decay widths. Their amplitudes are

$$T_{K' \rightarrow K^*\pi}^\mu = ig_{K' \rightarrow K^*\pi}(p+q)^\mu, \quad (205)$$

$$T_{K' \rightarrow K\rho}^\mu = ig_{K' \rightarrow K\rho}(p+q)^\mu, \quad (206)$$

here  $p$  is the momentum of  $K'$ ,  $q$  is the momentum of  $\pi$  ( $K$ ). The coupling constants are

$$g_{K' \rightarrow K^*\pi} = 4\mathcal{J}_{1,1}[\bar{f}'_K \bar{f}_{K^*} \bar{f}_\pi], \quad g_{K' \rightarrow K\rho} = 4\mathcal{J}_{1,1}[\bar{f}'_K \bar{f}_K \bar{f}_\rho]. \quad (207)$$

By calculating the integrals in the above formulae we have  $g_{K' \rightarrow K^*\pi} \approx -1.4$  and  $g_{K' \rightarrow K\rho} \approx -1.2$ . The decay widths thereby are

$$\Gamma_{K' \rightarrow K^*\pi} \approx 100 \text{ MeV}, \quad \Gamma_{K' \rightarrow K\rho} \approx 50 \text{ MeV}. \quad (208)$$

These processes have been observed in experiment and the decay widths are <sup>9</sup> [1]

$$\Gamma_{K' \rightarrow K^*\pi}^{exp} \sim 109 \text{ MeV}, \quad \Gamma_{K' \rightarrow K\rho}^{exp} \sim 34 \text{ MeV}. \quad (209)$$

The remaining decay  $K' \rightarrow K\pi\pi$  into three particles requires more complicated calculations. In this case, one must consider a box diagram, Fig.5.(a), and two types of diagrams, Fig.5.(b), with intermediate  $\sigma$ - and  $K_0^*$ -resonances. The diagrams for resonance channels are approximated by the relativistic Breit-Wigner function. The integration over the kinematically relevant range in the phase space for final states gives

$$\Gamma_{K' \rightarrow K\pi\pi} \sim 1 \text{ MeV}. \quad (210)$$

### 4.3 Strong decays of scalar mesons

The ground and excited states of scalar mesons  $f_0$ ,  $a_0$ , and  $K_0^*$  decay mostly into pairs of pseudoscalar mesons.

They can easily be related to  $Z_{\sigma(\varphi),i}^a$  introduced at the beginning of our paper.

All amplitudes that describe processes of the type  $\sigma \rightarrow \varphi_1\varphi_2$  can be divided into two parts:

$$\begin{aligned} T_{\sigma \rightarrow \varphi_1\varphi_2} &= C \left( -\frac{iN_c}{(2\pi)^4} \right) \int_{\Lambda_3} d^4k \frac{\text{Tr}[(m + \not{k} + \not{p}_1)\gamma_5(m + \not{k})\gamma_5(m + \not{k} - \not{p}_2)]}{(m^2 - k^2)(m^2 - (k + p_1)^2)(m^2 - (k - p_2)^2)} \\ &= 4mC \left( -\frac{iN_c}{(2\pi)^4} \right) \int_{\Lambda_3} d^4k \frac{\left[ 1 - \frac{p_1 \cdot p_2}{m^2 - k^2} \right]}{(m^2 - (k + p_1)^2)(m^2 - (k - p_2)^2)} \\ &= 4mC[I_2(m, p_1, p_2) - p_1 \cdot p_2 I_3(m, p_1, p_2)] = T^{(1)} + T^{(2)}. \end{aligned} \quad (211)$$

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<sup>9</sup>The accuracy of measurements carried out for the decays of  $K'$  is not given in [1].

Here  $C = 4g_\sigma g_{\varphi_1} g_{\varphi_2}$  and  $p_1, p_2$  are momenta of pseudoscalar mesons. We rewrite the amplitude  $T_{\sigma \rightarrow \varphi_1 \varphi_2}$  in another form

$$T_{\sigma \rightarrow \varphi_1 \varphi_2} \approx 4mZ^{-1/2} g_{\varphi_1} \left[ 1 - p_1 \cdot p_2 \frac{I_3(m)}{I_2(m)} \right], \quad (212)$$

$$p_1 \cdot p_2 = \frac{1}{2}(M_\sigma^2 - M_{\varphi_1}^2 - M_{\varphi_2}^2). \quad (213)$$

We assumed here that the  $I_3/I_2$  ratio slowly changes with the momentum in comparison with the factor  $p_1 \cdot p_2$ , therefore, we ignore their momentum dependence in (212). With this assumption we are going to obtain just a qualitative picture for decays of the excited scalar mesons.

In Eqs. (211) and (212), we omitted the contributions from the diagrams that include form factors at vertices. The whole set of diagrams consists of those containing zero, one, two, and three form factors. To obtain the complete amplitude, one must sum up all contributions.

After these general comments, let us consider the decays of  $a_0(1450)$ ,  $f_0(1370)$ ,  $f_J(1710)$ , and  $K_0^*(1430)$ . First, we estimate the decay width of the process  $\hat{a}_0 \rightarrow \eta\pi$ , taking the mixing coefficients from Tables 1 and 2 (see Appendix B for details). The result is

$$T_{\hat{a}_0 \rightarrow \eta\pi}^{(1)} \approx 0.2 \text{ GeV}, \quad T_{\hat{a}_0 \rightarrow \eta\pi}^{(2)} \approx 3.5 \text{ GeV}, \quad (214)$$

$$\Gamma_{\hat{a}_0 \rightarrow \eta\pi} \approx 160 \text{ MeV}. \quad (215)$$

From this calculation one can see that  $T^{(1)} \ll T^{(2)}$  and the amplitude is dominated by its second part,  $T^{(2)}$ , that is momentum-dependent. The first part is small because the diagrams with different numbers of form factors cancel each other. As a consequence, in all processes where an excited scalar meson decays into a pair of ground pseudoscalar states, the second part of the amplitude determines the rate of the process.

For the decay  $\hat{a}_0 \rightarrow \pi\eta'$  we obtain the amplitudes

$$T_{\hat{a}_0 \rightarrow \pi\eta'}^{(1)} \approx 0.8 \text{ GeV}, \quad T_{\hat{a}_0 \rightarrow \pi\eta'}^{(2)} \approx 3 \text{ GeV}, \quad (216)$$

and the decay width

$$\Gamma_{\hat{a}_0 \rightarrow \pi\eta'} \approx 36 \text{ MeV}. \quad (217)$$

The decay of  $\hat{a}_0$  into kaons is described by the amplitudes  $T_{\hat{a}_0 \rightarrow K^+ K^-}$  and  $T_{\hat{a}_0 \rightarrow \bar{K}^0 K^0}$  which, in accordance with our scheme, can again be divided into two parts:  $T^{(1)}$  and  $T^{(2)}$  (see Appendix B for details):

$$T_{\hat{a}_0 \rightarrow K^+ K^-}^{(1)} \approx 0.2 \text{ GeV}, \quad T_{\hat{a}_0 \rightarrow K^+ K^-}^{(2)} \approx 2.1 \text{ GeV} \quad (218)$$

and the decay width is

$$\Gamma_{\hat{a}_0 \rightarrow KK} = \Gamma_{\hat{a}_0 \rightarrow K^+ K^-} + \Gamma_{\hat{a}_0 \rightarrow \bar{K}^0 K^0} \approx 100 \text{ MeV}. \quad (219)$$

Qualitatively, our results do not contradict the experimental data.

$$\Gamma_{\hat{a}_0}^{\text{tot}} = 265 \pm 13 \text{ MeV}, \quad BR(\hat{a}_0 \rightarrow KK) : BR(\hat{a}_0 \rightarrow \pi\eta) = 0.88 \pm 0.23. \quad (220)$$

The decay widths of radial excitations of scalar isoscalar mesons are estimated in the same way as shown above:

$$\Gamma_{\hat{\sigma} \rightarrow \pi\pi} \approx \begin{cases} 550 \text{ MeV} (M_{\hat{\sigma}} = 1.3 \text{ GeV}) \\ 460 \text{ MeV} (M_{\hat{\sigma}} = 1.25 \text{ GeV}), \end{cases} \quad (221)$$

$$\Gamma_{\hat{\sigma} \rightarrow \eta\eta} \approx \begin{cases} 24 \text{ MeV} (M_{\sigma} = 1.3 \text{ GeV}) \\ 15 \text{ MeV} (M_{\sigma} = 1.25 \text{ GeV}), \end{cases} \quad (222)$$

$$\Gamma_{\hat{\sigma} \rightarrow \sigma\sigma} \approx \begin{cases} 6 \text{ MeV} (M_{\sigma} = 1.3 \text{ GeV}) \\ 5 \text{ MeV} (M_{\sigma} = 1.25 \text{ GeV}), \end{cases} \quad (223)$$

$$\Gamma_{\hat{\sigma} \rightarrow KK} \sim 5 \text{ MeV}, \quad (224)$$

$$\Gamma_{K_0^* \rightarrow K\pi} \approx 300 \text{ MeV}. \quad (225)$$

The heaviest scalar isoscalar meson in our model has the mass 1600 MeV (see Table 3) to be associated with an experimentally found meson state. From experimental data [1], we find two possible candidates for the role of a member of the radially excited meson nonet:  $f_0(1500)$  and  $f_0(1710)$ . The extra meson state can be explained by possible mixing of members of the  $\bar{q}q$  meson nonets with a gluon bound state, the glueball. Indeed, on the mass scale, both meson states lie in the region where the hypothetical glueball state is expected to exist. Insofar as we did not include the glueball into our model (however, we are going to do this in our further works), the picture is not complete. Nevertheless, we are free to make a hypothesis concerning the contents of  $f_0(1500)$  and  $f_0(1710)$ . We expect that one of these states is mostly a quarkonium with just a negligible admixture of the glueball state whereas the other is essentially mixed with the glueball. The mass splitting that always appears when two or more states mix with each other will either increase or decrease the mass of a quarkonium, depending on the mass of a “bare” (unmixed) glueball state either being smaller or greater than the mass of the quarkonium. After mixing we expect to find the  $\bar{q}q$  bound state with the mass 1500 MeV or 1710 MeV.

To decide which of them is the quarkonium with a small content of a glueball state, associated with the radial excitation of  $f_0(980)$ , we estimate its decay widths for two cases: first for the mass 1710 MeV quarkonium

$$\begin{aligned} \Gamma_{f_0(1710) \rightarrow 2\pi} &\approx 3 \text{ MeV}, & \Gamma_{f_0(1710) \rightarrow 2\eta} &\approx 40 \text{ MeV}, \\ \Gamma_{f_0(1710) \rightarrow \eta\eta'} &\approx 42 \text{ MeV}, & \Gamma_{f_0(1710) \rightarrow KK} &\approx 24 \text{ MeV}, \end{aligned} \quad (226)$$

and then for the mass 1500 MeV quarkonium

$$\begin{aligned} \Gamma_{f_0(1500) \rightarrow 2\pi} &\approx 3 \text{ MeV}, & \Gamma_{f_0(1500) \rightarrow 2\eta} &\approx 20 \text{ MeV}, \\ \Gamma_{f_0(1500) \rightarrow \eta\eta'} &\approx 10 \text{ MeV}, & \Gamma_{f_0(1500) \rightarrow KK} &\approx 20 \text{ MeV}. \end{aligned} \quad (227)$$

The decays of  $f_0(1500)$  and  $f_0(1710)$  into  $\sigma\sigma$  are negligible, so we disregard them. From the experimental data we have:

$$\Gamma_{\sigma'}^{\text{tot}} = 200 - 500 \text{ MeV}, \quad \Gamma_{f_0(1710)}^{\text{tot}} = 133 \pm 14 \text{ MeV}, \quad \Gamma_{f_0(1500)}^{\text{tot}} = 112 \pm 10 \text{ MeV}. \quad (228)$$

Thus, we can see that in the case of  $f_0(1500)$  being a  $\bar{q}q$  state there is a deficit in the decay widths whereas for  $f_J(1710)$  the result is close to experiment. From this we conclude that the meson  $f_J(1710)$  better suits for the role of a member of the  $\bar{q}q$  nonets as a radially excited partner for  $f_0(980)$  and the meson state  $f_0(1370)$  as the first radial excitation of  $f_0(400 - 1200)$ . As to  $f_0(1500)$ , the  $\bar{q}q$  model works bad for it. This gives us the idea that  $f_0(1500)$  is essentially mixed with the glueball state which significantly contributes to its decay width. Our interpretation of  $f_0(1500)$  and  $f_0(1710)$  is in agreement with other approaches where similar conclusions were made by the  $K$ -matrix method [36] and QCD sum rules [37].

The strong decay widths of ground states of scalar mesons were calculated in paper [32] in the framework of the standard NJL model with 't Hooft interaction where it was shown that a strange scalar meson state with a mass about 960 MeV decays into  $K\pi$  with the rate

$$\Gamma_{K_0^*(960) \rightarrow K\pi} = \frac{3}{Z\pi M_{K_0^*}^3} \left( \frac{m_u m_s}{2F_\pi} \right)^2 \Phi(M_{K_0^*}, M_K, M_\pi) \approx 360 \text{ MeV}. \quad (229)$$

By comparing this result with the analysis of phase shifts given in [5] where an evidence for existence of a scalar strange meson with the mass equal to  $905 \pm 50$  MeV and decay width  $545 \pm 170$  MeV is shown, we identify the state  $K_0^*(960)$  as a member of the ground scalar meson nonet. The state  $K_0^*(1430)$  is thereby its first radial excitation.

#### 4.4 Strong decays of $\eta(1295)$ and $\eta(1440)$ .

The mesons  $\eta(1295)$  and  $\eta(1440)$  have common decay modes:  $a_0\pi$ ,  $\eta\pi\pi$ ,  $\eta(\pi\pi)_{S\text{-wave}}$ ,  $K\bar{K}\pi$ , moreover, the heavier pseudoscalar  $\eta(1440)$  decays also into  $KK^*$ . For the processes with two secondary particles, the calculations of decay widths are done in the same way as shown in the previous subsection, by calculating the corresponding triangle diagrams.

Let us consider the decay  $\eta \rightarrow a_0\pi$ . The corresponding amplitude is of the same form as given in (211) for decays of the type  $\sigma \rightarrow \varphi\varphi$ . It can also be divided into two parts  $T^{(1)}$  and  $T^{(2)}$  which in our approximation are constant and momentum-dependent in the sense explained in the previous subsection (see (212) and the text below):

$$T_{\hat{\eta} \rightarrow a_0\pi}^{(1)} \approx 0.3 \text{ GeV}, \quad T_{\hat{\eta} \rightarrow a_0\pi}^{(2)} \approx -1 \text{ GeV} \quad (230)$$

Therefore, the decay width is

$$\Gamma_{\hat{\eta} \rightarrow a_0\pi} \approx 3 \text{ MeV}. \quad (231)$$

The decay  $\hat{\eta} \rightarrow \eta(\pi\pi)_{S\text{-wave}}$  is nothing else than the decay  $\hat{\eta} \rightarrow \eta\sigma \rightarrow \eta(\pi\pi)_{S\text{-wave}}$  where we have the  $\sigma$ -meson in the final state decaying then into pions in the S-wave. We simply calculate  $\hat{\eta} \rightarrow \eta\sigma$ , with  $\sigma$  as a decay product.

The calculation of decay widths for the rest of the decay modes with two particles in the final state is similar and the result is given in Table 4.

The decay  $\hat{\eta}' \rightarrow KK^*$  differs from the other modes due to the strange vector meson among the decay products. In this case we have

$$T_{\hat{\eta}' \rightarrow KK^*}^\mu = 4(p_1 + p_2)^\mu \left( [g_u g_K g_{K^*} I_2(m_u, m_s) + \dots] - \right.$$



Table 4:  $\eta(1295)$  and  $\eta(1440)$  decay modes.

	$a_0\pi$	$\eta\sigma$	$\eta\pi\pi$	$K\bar{K}\pi$	$KK^*$	$\Gamma^{\text{tot}}$
$\eta(1295)$	3 MeV	30 MeV	4 MeV	5 MeV	—	48 MeV
$\eta(1440)$	10 MeV	3 MeV	6 MeV	26 MeV	70 keV	45 MeV

$$\sqrt{2}[g_s g_K g_{K^*} I_2(m_u, m_s) + \dots]) \quad (232)$$

where  $p_1$  is the momentum of  $\hat{\eta}'$ ;  $p_2$ , the momentum of  $K$ ; and dots stand for the terms with form factors (not displayed here). These two parts are of the same order of magnitude and differ in sign and therefore cancel each other, which reduces the decay width up to tens of keV:

$$\Gamma_{\hat{\eta}' \rightarrow KK^*} \approx 70 \text{ keV}. \quad (233)$$

When there are three particles in the final state, poles appear in amplitudes, related to intermediate scalar resonances. As it is well known from  $\pi\pi$  scattering, these diagrams can play a crucial role in the description of such processes. So, in addition to the "box" diagram we take account of the diagrams with poles provided by  $\sigma$ ,  $f_0$ , and  $a_0$  resonances (see Fig. 6). Here we neglect the momentum dependence in the box diagram approximating it by a constant. The amplitude is thereby

$$\begin{aligned} T_{\hat{\eta} \rightarrow \eta\pi\pi} = B &+ \frac{c_{\sigma\eta\hat{\eta}} c_{\sigma\pi\pi}}{M_\sigma^2 - s - iM_\sigma\Gamma_\sigma} + \frac{c_{f_0\eta\hat{\eta}} c_{f_0\pi\pi}}{M_{f_0}^2 - s - iM_{f_0}\Gamma_{f_0}} \\ &+ \frac{c_{a_0\hat{\eta}\pi} c_{a_0\eta\pi}}{M_{a_0}^2 - t - iM_{a_0}\Gamma_{a_0}} + \frac{c_{a_0\hat{\eta}\pi} c_{a_0\eta\pi}}{M_{a_0}^2 - u - iM_{a_0}\Gamma_{a_0}} + \text{excited}, \end{aligned} \quad (234)$$

where  $B$  is given by the "box" diagram:

$$B = 12 \left( \frac{m_u}{F_\pi} \right)^2 Z^{-1} [R_{11} R_{12} + \dots] \quad (235)$$

where dots stand for the contribution from diagrams with form factors, and  $R_{ij}$  are taken from Table 2 (for  $\eta$  and  $\hat{\eta}$ ). The coefficients  $c_{\sigma\varphi\varphi}$  represent the amplitudes describing decays of a scalar to a couple of pseudoscalars; the calculation of them was discussed in the previous subsection. In general, they are momentum-dependent.

The kinematic invariants  $s$ ,  $t$ , and  $u$  are Mandelstam variables:  $s = (p_{\pi_1} + p_{\pi_2})^2$ ,  $t = (p_\eta + p_{\pi_1})^2$ ,  $u = (p_\eta + p_{\pi_2})^2$

The "excited" terms are contributions from excited scalar resonances of a structure similar to that for the ground states. The decay widths of processes  $\hat{\eta} \rightarrow \eta\pi\pi$  and  $\hat{\eta}' \rightarrow \eta\pi\pi$  are thereby

$$\Gamma_{\hat{\eta} \rightarrow \eta\pi\pi} \approx 4 \text{ MeV}, \quad \Gamma_{\hat{\eta}' \rightarrow \eta\pi\pi} \approx 6 \text{ MeV}. \quad (236)$$

For the processes  $\hat{\eta} \rightarrow K\bar{K}\pi$  and  $\hat{\eta}' \rightarrow K\bar{K}\pi$  we approximate their decay widths by neglecting the pole-diagram contribution because it turns out that the "box" is dominant here. The result is given in Table 4.

Unfortunately, the branching ratios for different decay modes of  $\eta(1295)$  and  $\eta(1440)$  are not well known from experiment; so one can only find their total decay widths

$$\Gamma_{\eta(1295)}^{\text{tot}} = 53 \pm 6 \text{ MeV}, \quad \Gamma_{\eta(1440)}^{\text{tot}} = 50 - 80 \text{ MeV}, \quad (237)$$

which is in satisfactory agreement with our results.

Strong and electromagnetic decays of the ground states of  $\hat{\eta}$  and  $\hat{\eta}'$  mesons were investigated within the framework of the standard NJL model in [9, 10] and we do not consider them here.

## 5. Conclusion

Let us summarize and discuss main features of the nonlocal NJL model proposed here and basic results obtained in our work.

A simple generalization of the NJL model to a nonlocal four-quark interaction of the separable type was suggested to describe first radial excitations of the scalar, pseudoscalar, and vector mesons. The nonlocality was introduced into quark currents by means of simple form factors, while preserving the local form of the ground and excited meson states. On the one hand, form factors can be written in a relativistic invariant form. On the other hand, the form factor parameters can be chosen so that the gap equations keep the conventional form, which leads to constant constituent quark masses and quark condensates. As a result, all low energy theorems are fulfilled in our model in the chiral limit (see Section 2). Therefore, the introduction of excited meson states does not destroy those attractive features which the NJL model is characteristic of.

The model contains six basic and seven additional form factor parameters. The basic ones are defined like in the standard (local) NJL model. They are the quark masses  $m_u = m_d$ ,  $m_s$ , the cut-off parameter  $\Lambda_3$ , and three quark coupling constants  $G$ ,  $G_V$ ,  $K$ . To determine them, we used six input quantities:  $F_\pi$ ,  $g_\rho$ ,  $M_\pi$ ,  $M_K$ ,  $M_\rho$ , and the mass difference  $M_\eta^2 - M_{\eta'}^2$ . Then, we predicted the masses of  $\eta$ ,  $\eta'$ ,  $K_0^*$ ,  $\varphi$  mesons and also the masses of the scalar and axial-vector meson nonets. The weak decay constant  $F_K$  and all strong coupling meson constants are calculated.

Upon the excited meson states are included, a great number of form factors appears in the model. They are necessary to describe radial excitations of the three meson nonets: scalar, pseudoscalar and vector. Each form factor contains two parameters: the external parameter  $c_{qq}^U$  characterizing to what extent the interaction of excited states is stronger than that of the ground ones and the internal (slope) parameter  $d_{qq}$  determining the shape of the wave function of an excited meson state.

We give an unambiguous definition of the slope parameters for scalar mesons from the condition that the excited states do not contribute to quark condensates. Then, we assume the slope parameters to be the same for any sort of meson fields. Moreover, in favor of the global chiral symmetry, we put the scalar meson form factors equal to the pseudoscalar meson ones. As a result, only seven independent parameters are left:  $c_{uu}^\pi$ ,  $c_{us}^K$ ,  $c_{uu}^{\eta,\eta'}$ ,  $c_{ss}^{\eta,\eta'}$ ,  $c_{uu}^{\rho,\omega}$ ,  $c_{us}^{K_0^*}$ ,

$c_{ss}^\varphi$ . They are fixed by masses of radially excited pseudoscalar and vector mesons. When this procedure is completed, we are able to predict the masses of scalar mesons and identify them with experimentally observed meson states.

The major results obtained in our work are:

- 1) A nonlocal chiral quark model with a quark interaction of the separable type was developed to describe the ground and first radially excited states of mesons represented by local fields. In this model, the quark condensate and gap equations are conserved in the standard form, and all low-energy theorems are fulfilled.
- 2) In a realistic  $U(3) \times U(3)$  version of the model, the  $U_A(1)$  problem is solved by introducing the 't Hooft interaction. The mixing of pseudoscalar isoscalar meson states, the ground  $\eta$ ,  $\eta'$ , and the radially excited  $\hat{\eta}$ ,  $\hat{\eta}'$ , due to the 't Hooft interaction, was taken into account.
- 3) In the framework of the proposed model, a satisfactory description of the masses of ground and first radially excited pseudoscalar and vector meson states was obtained.
- 4) The mass spectrum for scalar meson nonets (ground and first radially excited) is predicted on the basis of the proposed model and with the assumption on the form factors, based on the global chiral symmetry, that the form factors for scalar mesons are the same as for the pseudoscalars ones.
- 5) The members of quark-antiquark nonets, whose physics the proposed model is intended to describe, are identified with twenty seven physically observed scalar, pseudoscalar, and vector meson states,
- 6) The weak decay constants  $F_{\pi'}$ ,  $F_K$ , and  $F_{K'}$  are estimated.
- 7) The widths of main strong decays of radially excited scalar, pseudoscalar, and vector meson nonets are estimated. The results are in satisfactory agreement with experimental data.

Let us make some comments on the identification of the meson nonets' members. While it seems clear how to identify the members of pseudoscalar and vector meson nonets, the scalar mesons require more words to say. From our calculations we come to the following interpretation of  $f_0(1370)$ ,  $f_J(1710)$ ,  $a_0(1470)$ ,  $K_0^*(1430)$  mesons: we consider them as the first radial excitations of the ground states  $f_0(400 - 1200)$ ,  $f_0(980)$ ,  $a_0(980)$  and  $K_0^*(960)$ <sup>10</sup>.

In this picture, however, no place is reserved for the  $f_0(1500)$  meson. To include it, we need an additional meson state in our model that is not a bound  $\bar{q}q$ -system (there is no vacancy in the considered multiplets) but rather it is a bound colorless gluon state [38]. There are many reasons that the state  $f_0(1500)$  is essentially mixed with a glueball [36, 37]. However, in this

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<sup>10</sup> The light strange scalar of a mass about 900 MeV is not included into the summary tables of PDG [1]. However, there are evidences from the phase shift analysis [5] that a state (known as  $\kappa(900)$ ) with the mass 950 MeV does exist.

paper we did not take the glueball into account. Therefore, we cannot say how much it can affect  $\bar{q}q$  meson states. However, we are going to tackle this problem in our further work. In the present paper, we obtain a bound quark-antiquark state with the mass about 1600 MeV, so we have to decide which of the observed meson states,  $f_0(1500)$  or  $f_J(1710)$ , is to be associated with this member of the nonet of the radially excited scalar mesons in our model. We have chosen  $f_J(1710)$ . The reason for this choice is based both upon the results obtained in ref. [36, 37] and on our estimates of the decay widths discussed in Section 4.

In conclusion, we would like to outline further steps to improve our model. First of all, a glueball state can be included into the effective Lagrangian. This will allow us to correct the description of the scalar states  $f_0(980)$ ,  $f_0(1370)$ ,  $f_0(1710)$  and include  $f_0(1500)$  (presumed to be essentially mixed with a glueball) into the whole picture. The mixing of all the states will play an important role in this case. By now, we took account only of the mixing among  $f_0(400 - 1200)$ ,  $f_0(980)$ ,  $f_0(1370)$ ,  $f_0(1710)$  and among  $\eta$ ,  $\eta'$ ,  $\hat{\eta}$ ,  $\hat{\eta}'$ . Nevertheless, our investigation revealed that the meson states  $\eta(1300)$ ,  $\eta(1470)$ ,  $f_0(1370)$ ,  $a_0(1470)$ ,  $f_0(1710)$ ,  $K_0^*(1430)$  are the first radial excitations of  $\eta(590)$ ,  $\eta(950)$ ,  $f_0(400 - 1200)$ ,  $a_0(980)$ ,  $f_0(980)$ ,  $K_0^*(960)$ .

Second, the absence of quark confinement is still a common flaw of NJL-like models with a local quark interaction. There are several approaches suggested to find a solution of this problem. Among them there are various potential models, models where the pole in the quark propagator is excluded [39], *etc.* We are going to continue to work with our own approach which was suggested in ref. [40].

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## Appendix

### A Coefficients of the free part of the effective Lagrangian for scalar isoscalar mesons.

The functions  $K_{\sigma(\varphi),ij}^{[a,b]}$  introduced in Sec. 3 of Chap. 4 (131) are defined as follows

$$\begin{aligned}
K_{\sigma(\varphi),11}^{[a,a]}(P) &= Z_{\sigma(\varphi),1}^a(P^2 - (m_q^a \pm m_{q'}^a)^2 - M_{\sigma^a(\varphi^a),1}^2), \\
K_{\sigma(\varphi),22}^{[a,a]}(P) &= Z_{\sigma(\varphi),2}^a(P^2 - (m_q^a \pm m_{q'}^a)^2 - M_{\sigma^a(\varphi^a),2}^2), \\
K_{\sigma(\varphi),12}^{[a,a]}(P) &= K_{\sigma(\varphi),21}^{[a,a]}(P) = \gamma_{\sigma(\varphi)}^a(P^2 - (m_q^a \pm m_{q'}^a)^2), \\
K_{\sigma(\varphi),11}^{[8,9]}(P) &= K_{\sigma(\varphi),11}^{[9,8]}(P) = \frac{1}{2} \left( T^{S(P)} \right)_{89}^{-1}, \\
K_{\sigma(\varphi),12}^{[8,9]}(P) &= K_{\sigma(\varphi),12}^{[9,8]}(P) = K_{\sigma(\varphi),21}^{[8,9]}(P) = 0, \\
K_{\sigma(\varphi),21}^{[9,8]}(P) &= K_{\sigma(\varphi),22}^{[8,9]}(P) = K_{\sigma(\varphi),22}^{[9,8]}(P) = 0
\end{aligned} \tag{238}$$

where the “bare” meson masses are

$$\begin{aligned}
M_{\sigma^8(\varphi^8),1}^2 &= (Z_{\sigma(\varphi),1}^8)^{-1} \left( \frac{1}{2} (T^{S(P)})_{88}^{-1} - 8I_1(m_u) \right), \\
M_{\sigma^9(\varphi^9),1}^2 &= (Z_{\sigma(\varphi),1}^9)^{-1} \left( \frac{1}{2} (T^{S(P)})_{99}^{-1} - 8I_1(m_s) \right), \\
M_{\sigma^8(\varphi^8),2}^2 &= (Z_{\sigma(\varphi),2}^8)^{-1} \left( \frac{1}{2G} - 8I_1^{ff}(m_u) \right), \\
M_{\sigma^9(\varphi^9),2}^2 &= (Z_{\sigma(\varphi),2}^9)^{-1} \left( \frac{1}{2G} - 8I_1^{ff}(m_s) \right).
\end{aligned} \tag{239}$$

In the case of isoscalar mesons it is convenient to combine the scalar and pseudoscalar fields into 4-vectors

$$\Phi = (\varphi_1^{8r}, \varphi_2^{8r}, \varphi_1^{9r}, \varphi_2^{9r}), \quad \Sigma = (\sigma_1^{8r}, \sigma_2^{8r}, \sigma_1^{9r}, \sigma_2^{9r}), \tag{240}$$

and introduce  $4 \times 4$  matrix functions  $\mathcal{K}_{\sigma(\varphi),ij}$ , instead of old  $K_{\sigma(\varphi),ij}^{[a,b]}$ , where indices  $i, j$  run from 1 through 4. This allows us to rewrite the free part of the effective Lagrangian which then, with the meson fields renormalized, looks as follows

$$\mathcal{L}^{(2)}(\Sigma, \Phi) = \frac{1}{2} \sum_{i,j=1}^4 (\Sigma_i \mathcal{K}_{\sigma(\varphi),ij}(P) \Sigma_j + \Phi_i \mathcal{K}_{\varphi(\varphi),ij}(P) \Phi_j). \tag{241}$$

and the functions  $\mathcal{K}_{\sigma(\varphi),ij}$  are

$$\begin{aligned}
\mathcal{K}_{\sigma(\varphi),11}(P) &= P^2 - (m_u \pm m_u)^2 - M_{\sigma^8(\varphi^8),1}^2, \\
\mathcal{K}_{\sigma(\varphi),22}(P) &= P^2 - (m_u \pm m_u)^2 - M_{\sigma^8(\varphi^8),2}^2, \\
\mathcal{K}_{\sigma(\varphi),33}(P) &= P^2 - (m_s \pm m_s)^2 - M_{\sigma^9(\varphi^9),1}^2, \\
\mathcal{K}_{\sigma(\varphi),44}(P) &= P^2 - (m_s \pm m_s)^2 - M_{\sigma^9(\varphi^9),2}^2, \\
\mathcal{K}_{\sigma(\varphi),12}(P) &= \mathcal{K}_{\sigma(\varphi),21}(P) = \Gamma_{\sigma_u(\eta_u)}(P^2 - (m_u \pm m_u)^2), \\
\mathcal{K}_{\sigma(\varphi),34}(P) &= \mathcal{K}_{\sigma(\varphi),43}(P) = \Gamma_{\sigma_s(\eta_s)}(P^2 - (m_s \pm m_s)^2), \\
\mathcal{K}_{\sigma(\varphi),13}(P) &= \mathcal{K}_{\sigma(\varphi),31}(P) = (Z_{\sigma(\varphi),1}^8 Z_{\sigma(\varphi),2}^9)^{-1/2} (T^{S(P)})_{89}^{-1}.
\end{aligned} \tag{242}$$

Now, to transform (241) to the conventional form, one should just diagonalize a 4-dimensional matrix, which is better to do numerically.

## B The calculation of the amplitudes for decays of the excited scalar meson $\hat{a}_0$

Here we collect some instructive formulae that display a part of the details of calculations made in this work. Let us demonstrate how the amplitude of the decay  $\hat{a}_0 \rightarrow \eta\pi$  is obtained. The mixing coefficients are taken from Table 1. Moreover, the diagrams where pion vertices contain form factors are neglected because, as one can see from Table 1, their contribution is significantly reduced:

$$\begin{aligned}
T_{\hat{a}_0 \rightarrow \eta\pi}^{(1)} &= 4 \frac{m_u^2}{F_\pi} \left\{ 0.82 \cdot 0.71 \cdot Z^{-1/2} \frac{I_2(m_u)}{I_2(m_u)} - \right. \\
&\quad \left( 1.17 \cdot 0.71 \cdot Z^{-1/2} - 0.82 \cdot 0.11 \right) \frac{I_2^f(m_u)}{\sqrt{I_2(m_u) I_2^{ff}(m_u)}} - \\
&\quad \left. 1.17 \cdot 0.11 \cdot \frac{I_2^{ff}(m_u)}{I_2^{ff}(m_u)} \right\} \approx 0.2 \text{ GeV},
\end{aligned} \tag{243}$$

$$\begin{aligned}
T_{\hat{a}_0 \rightarrow \eta\pi}^{(2)} &= 2 \frac{m_u^2}{F_\pi} (M_{a_0}^2 - M_\eta^2 - M_\pi^2) \left\{ 0.82 \cdot 0.71 Z^{-1/2} \frac{I_3(m_u)}{I_2(m_u)} - \right. \\
&\quad \left( 1.17 \cdot 0.71 \cdot Z^{-1/2} - 0.82 \cdot 0.11 \right) \frac{I_3^f(m_u)}{\sqrt{I_2(m_u) I_2^{ff}(m)}} - \\
&\quad \left. 1.17 \cdot 0.11 \frac{I_3^{ff}(m_u)}{I_2(m_u)} \right\} \approx 3.5 \text{ GeV}.
\end{aligned} \tag{244}$$

The decay width thereby is

$$\Gamma_{\hat{a}_0 \rightarrow \eta\pi} = \frac{|T_{\hat{a}_0 \rightarrow \eta\pi}|^2}{16\pi M_{\hat{a}_0}^3} \sqrt{M_{\hat{a}_0}^4 + M_\eta^4 + M_\pi^4 - 2(M_{\hat{a}_0}^2 M_\eta^2 + M_{\hat{a}_0}^2 M_\pi^2 + M_\eta^2 M_\pi^2)} \approx 160 \text{ MeV}. \tag{245}$$

Here  $I_2(m_u) = 0.04$ ,  $I_2^f(m_u) = 0.014c$ ,  $I_2^{ff}(m_u) = 0.015c^2$ ,  $I_3(m_u) = 0.11 \text{ GeV}^{-2}$ ,  $I_3^f(m_u) = 0.07c \text{ GeV}^{-2}$ ,  $I_3^{ff}(m_u) = 0.06c^2 \text{ GeV}^{-2}$  and  $c$  is the external form factor parameter factored out and cancelled in the ratios of integrals.

For the decay into strange mesons we obtain (see Fig.1)

$$\begin{aligned}
T_{\hat{a}_0 \rightarrow K^+ K^-} &= C_K \left( -\frac{iN_c}{16\pi^2} \right) \int d^4k \frac{\text{Tr}[(m_u + \not{k} + \not{p}_1) \gamma_5 (m_s + \not{k}) \gamma_5 (m_u + \not{k} - \not{p}_2)]}{(m_s^2 - k^2)(m_u^2 - (\not{k} - \not{p}_1)^2)(m_u^2 - (\not{k} - \not{p}_2)^2)} \approx \\
&2C_K \left\{ (m_s + m_u) I_2(m_u) - \Delta I_2(m_u, m_s) - [m_s (M_{\hat{a}_0}^2 - 2M_K^2) - \right. \\
&\quad \left. 2\Delta^3] I_3(m_u, m_s) \right\},
\end{aligned} \tag{246}$$

where  $\Delta = m_s - m_u$  and

$$I_3(m_u, m_s) = -i \frac{N_c}{(2\pi)^4} \int_{\Lambda_3} \frac{d^4 k}{(m_u^2 - k^2)^2 (m_s^2 - k^2)}. \quad (247)$$

The coefficient  $C_K$  absorbs the Yukawa coupling constants and some structure coefficients. The integral  $I_2(m_u, m_s)$  is defined by (102). This is only the part of the amplitude without form factors. The complete amplitude of this process is a sum of contributions which contain also the integrals  $I_2^{f..f}$  and  $I_3^{f..f}$  with form factors. Thus, the amplitude is

$$T_{\hat{a}_0 \rightarrow K^+ K^-} = T^{(1)} + T^{(2)}, \quad (248)$$

$$T^{(1)} = \frac{m_u + m_s}{2F_K} \{(m_s + m_u) \cdot 0.13 - \Delta \cdot 0.21\} \approx 0.2 \text{ GeV}, \quad (249)$$

$$T^{(2)} = \frac{m_u + m_s}{2F_K} \{[m_s(M_{a_0}^2 - 2M_K^2) - 2\Delta^3] \cdot 1 \text{ GeV}^{-2}\} \approx 2.3 \text{ GeV}, \quad (250)$$

$$F_K = 1.2F_\pi.$$

The decay width therefore is evaluated to be

$$\Gamma_{\hat{a}_0 \rightarrow K^+ K^-} = \Gamma_{\hat{a}_0 \rightarrow \bar{K}^0 K^0} \approx 50 \text{ MeV}. \quad (251)$$

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## Figure captions

1. The quark loop contribution to the quadratic form  $K_{ij}(P)$ , eq.(94), of the effective action for  $\pi_1$ - and  $\pi_2$ -fields. Solid lines denote the NJL quark propagator. The  $\pi_1$ -field couples to quarks through a local vertex; the  $\pi_2$ -field, through the form factor,  $f(k_\perp)$ , marked by letter **f**.
2. The axial current of  $\pi_1$ - and  $\pi_2$ -fields, Eq.(44), as it follows from the Noether theorem. The cross denotes a local axial current of quark fields to which  $\pi_1$ - and  $\pi_2$ -fields couple through quark loops. The notation is the same as in Fig. 1.
3. Triangle diagrams describing decays of a  $\rho$ -meson. Each letter in a diagram indicates the presence of a form factor at a vertex.
4. Diagrams describing meson decays of the  $1 \rightarrow 2$  type.
5. Diagrams describing the decay  $K' \rightarrow K\pi\pi$ .
6. Diagrams describing the decay  $\hat{\eta} \rightarrow \eta\pi\pi$ . The black box stands for the sum of “box” diagrams represented by one-loop quark graphs with four meson vertices. The rest of the diagrams is a set of pole graphs with  $\sigma$ ,  $f_0$ , and  $a_0$  scalar resonances. The diagram with  $a_0$  is to be taken into account for two channels (due to the exchange of pions momenta). There are analogous contributions from radially excited resonances.

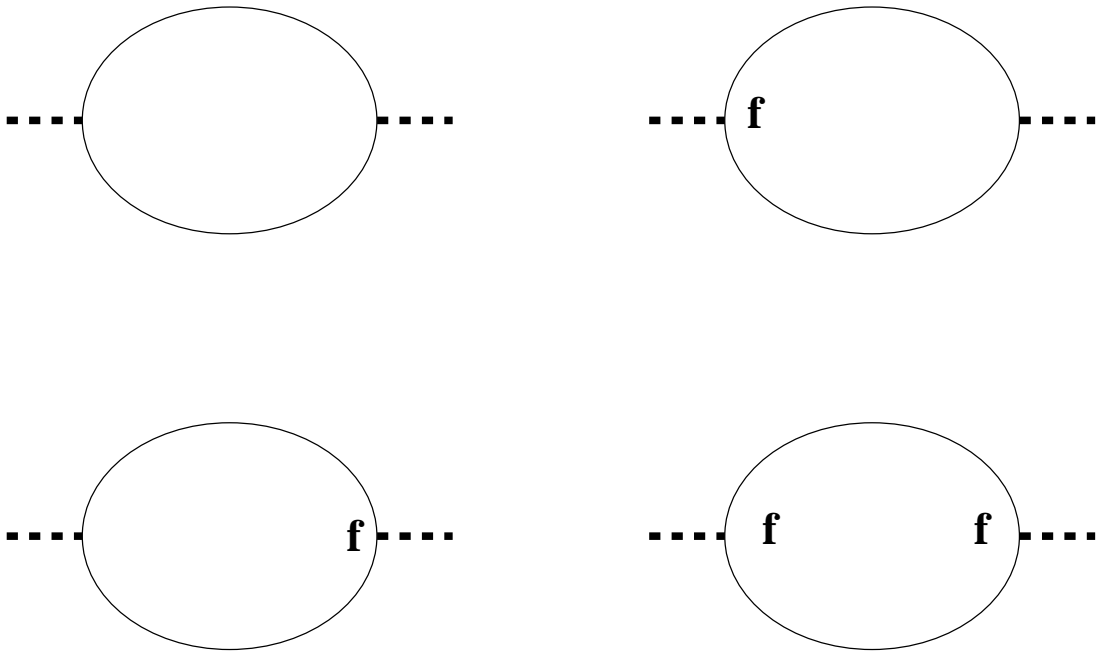


Figure 1:

$$\pi_1 \times \partial_\mu \cdots \bigcirc \times + \pi_2 \times \partial_\mu \cdots \mathbf{f} \times$$

Figure 2:

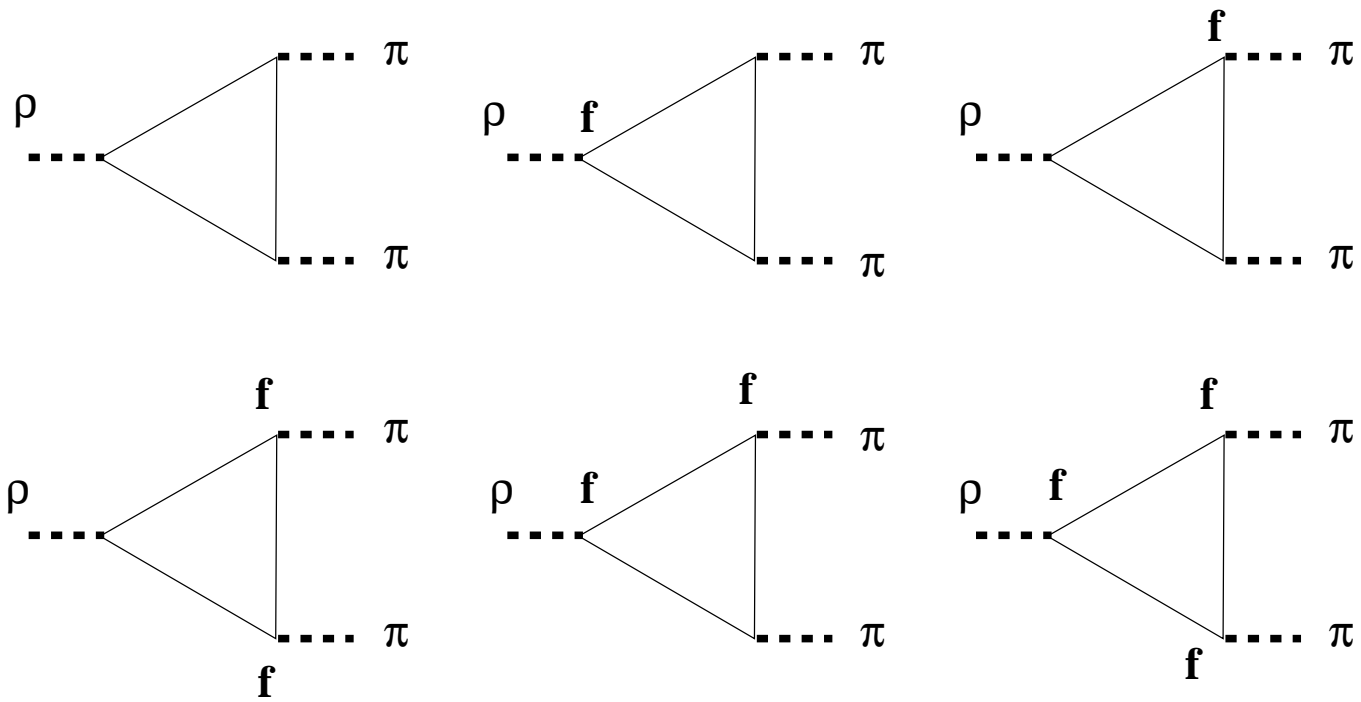


Figure 3:

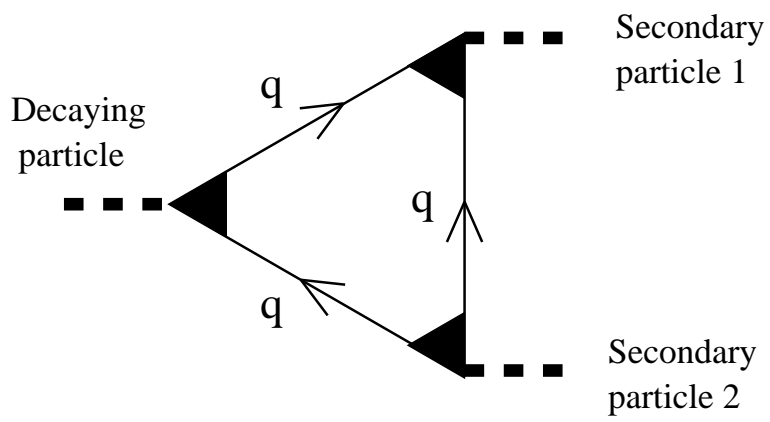


Figure 4:

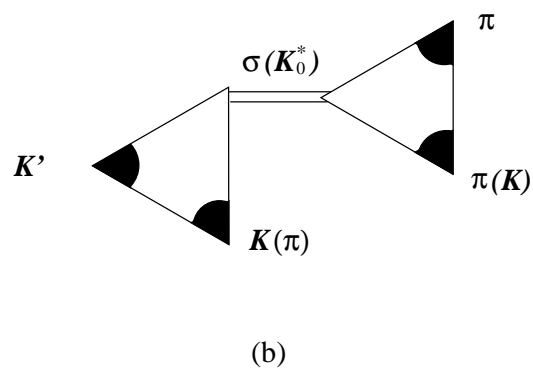
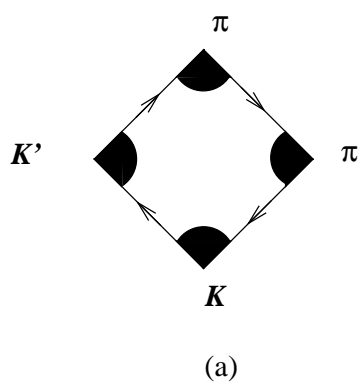


Figure 5:



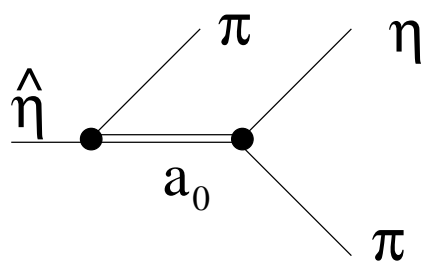
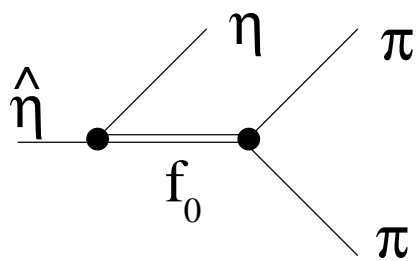
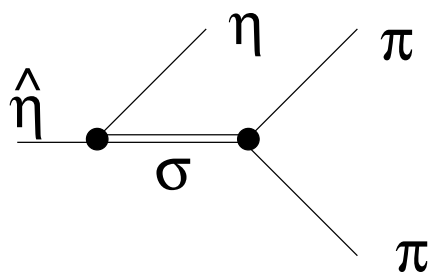
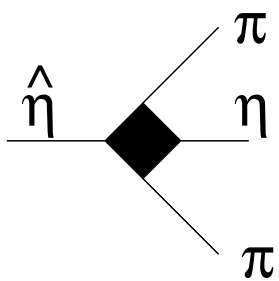


Figure 6: